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Topics in Graph Colouring and Extremal Graph Theory

Carl Feghali

A Thesis presented for the degree of
Doctor of Philosophy

School of Engineering and Computing Sciences
Durham University
England

October 2016

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Abstract

In this thesis we consider three problems related to colourings of graphs and one problem in extremal graph theory. Let G be a connected graph with n vertices and maximum degree $\Delta(G)$. Let $R_k(G)$ denote the graph with vertex set all proper k-colourings of G and two k-colourings are joined by an edge if they differ on the colour of exactly one vertex. Our first main result states that $R_{\Delta(G)+1}(G)$ has a unique non-trivial component with diameter $O(n^2)$. This result can be viewed as a reconfigurations analogue of Brooks' Theorem and completes the study of reconfigurations of colourings of graphs with bounded maximum degree.

A Kempe change is the operation of swapping some colours a, b of a component of the subgraph induced by vertices with colour a or b. Two colourings are Kempe equivalent if one can be obtained from the other by a sequence of Kempe changes. Our second main result states that all $\Delta(G)$-colourings of a graph G are Kempe equivalent unless G is the complete graph or the triangular prism. This settles a conjecture of Mohar (2007).

Motivated by finding an algorithmic version of a structure theorem for bull-free graphs due to Chudnovsky (2012), we consider the computational complexity of deciding if the vertices of a graph can be partitioned into two parts such that one part is triangle-free and the other part is a collection of complete graphs. We show that this problem is NP-complete when restricted to five classes of graphs (including bull-free graphs) while polynomial-time solvable for the class of cographs.

Finally we consider a graph-theoretic version formulated by Holroyd, Spencer and

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Talbot (2007) of the famous Erdős-Ko-Rado Theorem in extremal combinatorics and obtain some results for the class of trees.

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Declaration

The work in this thesis is based on research carried out at the School of Engineering and Computing Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Acknowledgements

I gratefully acknowledge the financial support by the Durham Doctoral Scholarship award without which the work in this thesis would not have been possible. I also acknowledge the excellent facilities provided by the School of Engineering and Computing Sciences department at Durham University.

I am greatly indebted to my first supervisor Matthew Johnson for his encouragements, patience and his outstanding supervisory framework which included productive meetings, the liberty for exploring new research areas and making collaboration with other researchers possible. There is no doubt that Matthew will make a lasting impression throughout my academic experience. I am also happy to thank my second supervisor Daniël Paulusma for his guidance, the numerous stimulating discussions and for acting as an excellent first supervisor when Matthew was on research leave. Daniël and Matthew have also provided generous comments on drafts of this thesis and acted as friends at times of doubt or stress, for which I am grateful.

Thanks are due to my co-authors for making collaboration a pleasurable experience. I would also like to thank Faisal N. Abu-Khzam and Haiko Müller for their guidance and helpful advice.

I am indebted to my family, especially to my father and mother, for their encouragements and support throughout my university studies.

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Chapter 1

Introduction

Graph theory is a branch of mathematics concerned with the study of graphs. A graph is a network: it consists of *nodes* or *vertices* and *lines* or *edges* each joining some pair of vertices. The origins of the theory can be traced back to 1736 when the mathematician Leonhard Euler gave a simple negative answer to the well-known Königsberg bridge problem; the problem of finding a walk that crosses each of the seven bridges of the city exactly once. (It seems almost inevitable that graphs were conceived in a practical setting: let vertices represent real-world objects and edges correspond to relations between pairs of these objects.)

In general there are numerous problems in practice that can be formulated and solved as problems on graphs (see [58] for a recent survey on applications of graph theory). The study of graphs is, however, also interesting in its own right and the emphasis in this work is on theoretical results. Let us nevertheless illustrate a folklore practical problem whose graph-theoretic formulation requires a specific notion – *graph colouring* – that is recurrent in this thesis:

The students at a university have a number of examinations at the end of each term. The problem then is to determine the minimum number of time slots to be scheduled for the examinations. Of course two examinations attended by the same student cannot be given the same time slot. Let us describe another way to think of this problem. A colouring of a graph is an assignment of colours to its vertices such that no two vertices joined by an edge receive the same colour. If the vertices are the examinations, and two vertices are joined by an edge whenever a student

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must attend both corresponding examinations, then the problem may be re-stated: determine the minimum number of colours to colour this graph.

It turns out that no efficient algorithm is known or likely to exist for solving this problem or several others [63]. (In fact the literature on problems unlikely to be solvable in polynomial time is immense.) One common way to circumvent this obstacle is to restrict our attention to special classes of graphs. This will allow us to exploit the structural properties of the graphs under consideration and thus, it is hoped, obtain polynomial-time algorithms.

Special graph classes are also useful in a more general setting: to prove or disprove a conjecture about a large (and possibly arbitrary) class of graphs we first establish results on a more restricted class and then try to generalise to the original class.

As we shall see in later chapters this approach for tackling a difficult problem is extensively used in the thesis. For a discussion and survey of various aspects related to special graph classes we refer the reader to some classical textbooks [21, 64].

The rest of this chapter is organised as follows. In Section 1.1, we give the basic definitions that are used throughout the thesis. The final section, Section 1.2, briefly introduces the topics explored and highlights our main contributions – proofs and extensive literature reviews are deferred until subsequent chapters.

1.1 Basic Definitions

A *graph* G is a pair of disjoint sets, $V(G)$ and $E(G)$, such that $V(G) \neq \emptyset$ and $E(G)$ is a set of unordered pairs $\{u, v\}$ of elements $u, v \in V(G)$. The set $V(G)$ is the set of *vertices* of G and the set $E(G)$ is the set of *edges*. Let $e = \{u, v\} \in E(G)$. Then $\{u, v\}$ is abbreviated uv. The vertices u and v are also called the *endvertices* of e and are said to be *adjacent* and *incident with* e. We assume G has no *loops* (edges with only one endvertex) or *multiple edges* (distinct edges having the same pair of endvertices) and $V(G)$ is finite. When the sets $V(G)$ and $E(G)$ are clear from the context, we shall write V for $V(G)$ and E for $E(G)$. The *order* of a graph G is defined as $|V(G)|$. Similarly the *size* of a graph G is defined as $|E(G)|$.

The *degree* of a vertex v, denoted $deg(v)$, is the number of vertices adjacent to

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v. Let $\Delta(G)$ or, if clear from the context, simply Δ denote the *maximum degree* of G, that is, $\Delta = \max\{deg(v) : v \in V\}$. The *minimum degree* of a graph G, denoted $\delta(G)$ or δ, is defined analogously. If $\Delta = \delta$, that is, every vertex of G has degree precisely Δ, then G is said to be Δ-regular. A 3-regular graph is called *cubic*.

A graph $H = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ and, equivalently, G is a *supergraph* of H, if $V' \subset V$ and $E' \subset E$. if in addition $E' = (V' \times V') \cap E$, then H is said to be an *induced subgraph* of G and we write $H = G[V']$. If $U \subset V$, then $G - U = G[V \setminus U]$ and if $U = \{u\}$ we usually write $G - u$ instead of $G - \{u\}$. A graph G *contains* a graph H if H is an induced subgraph of G. Two graphs are said to be *isomorphic* if there exists a $1 - 1$ correspondence between their vertex sets that preserves adjacency. We do not distinguish between isomorphic graphs. Given graphs G and H, G is said H-free if G does not contain an induced subgraph isomorphic to H. If $\mathcal{H}$ is a family of graphs, then G is $\mathcal{H}$-free if G is H-free for every graph $H \in \mathcal{H}$.

A *path* on n vertices $x_1, x_2, \ldots, x_n$ is denoted P_n and has edge set $x_i x_{i+1}$, $i = 1, \ldots, n-1$, and $x_j \neq x_k$ for any distinct indices j, k. The *length* of P_n is its size (that is, is equal to $n - 1$) and P_n is said to *join* the vertices x_1 and x_n. The *distance* between two distinct vertices u, v, denoted $d(u, v)$, is the minimum length over all paths joining u and v. The *diameter* of a graph G is defined as $\max\{d(u, v) : u, v \in V, u \neq v\}$. A *cycle* on n vertices $x_1, \ldots, x_n$ is denoted C_n and has edge set $\{x_1 x_n\} \cup \{x_i x_{i+1} : i = 1, \ldots, n-1\}$ and $x_j \neq x_k$ for any distinct indices j, k. The length of C_n is its size or, equivalently, its order. A graph on n vertices in which every pair of distinct vertices are adjacent is denoted K_n and is called *complete* or a *clique*. A set of vertices such that no two vertices of the set are adjacent is called an *independent* or a *stable* set.

A graph is *connected* if every pair of vertices of the graph is joined by a path. A graph that is not connected is called *disconnected*. A *component* of a graph is a maximal connected subgraph of the graph. Clearly a graph is connected if and only if the graph has exactly one component. The (disjoint) union $Q = G \cup H$ of vertex disjoint graphs G and H has as vertex set $V(Q) = V(G) \cup V(H)$ and edge set $E(Q) = E(G) \cup E(H)$. Thus a graph is disconnected if and only if it can be

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expressed as a disjoint union of at least two graphs. We *identify* two vertices x and y in a graph G if we replace them by a new vertex adjacent to all neighbours of x and y in G. The complement $\overline{G}$ of graph G has vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{xy : xy \notin E(G)\}$. Let v be a vertex of a graph G. Then v is called *universal* if $deg(v) = |V| - 1$ and *isolated* if $deg(v) = 0$.

A *k-colouring* of a graph G is a mapping $\phi : V \to \{1, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ if $uv \in E$. We call $\{1, \ldots, k\}$ the set of colours and refer to $\phi(u)$ as the colour of the vertex u. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that G has a k-colouring. Let $\{\mathcal{P}_1, \ldots, \mathcal{P}_q\}$ be a collection of graph properties. A *vertex partitioning* (also known as *decomposition*) of a graph G is a partition $V_1, \ldots, V_q$ of V into q parts such that, for $i = 1, \ldots, q$, the subgraph of G induced by vertices V_i satisfies property $\mathcal{P}_i$. Notice that a q-colouring of G corresponds to a vertex partitioning of G in which each V_i, $i = 1, \ldots, q$, induces an independent set.

1.2 Overview of Thesis

Let us first mention a fundamental result in graph colouring due to Brooks [24] that is relevant to the immediately following two chapters. The theorem draws a connection between the chromatic number and the maximum degree of a graph.

Brooks' Theorem. *Let G be a connected graph with maximum degree $\Delta \geq 1$. If G is not $K_{\Delta+1}$ or, if n is odd, C_n, then G has a Δ-colouring.*

In Chapter 2 we show an analogue of Brooks' Theorem in the setting of reconfigurations of colourings. The *k-colouring reconfiguration graph* of G, denoted $R_k(G)$, has as its vertex set all possible k-colourings of G, and two k-colourings γ_1 and γ_2 are joined by an edge if, for some vertex $u \in V$, $\gamma_1(u) \neq \gamma_2(u)$, and, for all $v \in V \setminus \{u\}$, $\gamma_1(v) = \gamma_2(v)$; that is, if γ_1 and γ_2 *disagree* on exactly one vertex. A k-colouring γ of a graph is *frozen* if, for every vertex v, every colour except $\gamma(v)$ is used on the neighbours of v. The length of a shortest path between colourings α and β in $R_k(G)$ is denoted $d_k(\alpha, \beta)$.

Theorem (Theorem 2.1). *Let G be a connected graph on n vertices with maximum*

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degree $\Delta \geq 1$, and let $k \geq \Delta + 1$. Let α be a k-colouring of G. If α is not frozen and G is not $K_{\Delta+1}$ or, if n is odd, C_n, then there exists a Δ-colouring γ of G such that $d_k(\alpha, \gamma)$ is $O(n^2)$.

We use the above theorem to prove that for $\Delta \geq 3$, $R_{\Delta+1}(G)$ consists of isolated vertices and at most one further component which has diameter $O(n^2)$. This result enables us to complete both a structural characterization and an algorithmic characterization for reconfigurations of colourings of graphs of bounded maximum degree. Chapter 2 is based on published joint work with Matthew Johnson and Daniël Paulusma [51, 54].

In Chapter 3 we address a conjecture of Mohar [97] which can again be viewed as an analogue of Brooks' Theorem in the setting of Kempe equivalence of colourings. Given a proper vertex colouring of G, a *Kempe chain* is a subset of V that induces a maximal connected subgraph of G in which every vertex has one of two colours. To make a *Kempe change* is to obtain one colouring from another by exchanging the colours of vertices in a Kempe chain. Two colourings are *Kempe equivalent* if each can be obtained from the other by a series of Kempe changes. Let $C_k(G)$ be the set of all k-colourings of G. The equivalence classes $C_k(G)/\sim_k$ are called *Kempe classes*. Notice that reconfigurations of colourings is a special case of Kempe equivalence in which one colouring can be transformed into another by a single vertex recolouring (note that this is s essentially a Kempe change of a Kempe chain consisting of one vertex only). The conjecture asserts that, for $k \geq 3$, all k-colourings of a k-regular graph that is not complete are Kempe equivalent. In the next theorem we completely settle the conjecture. (Note that, for every connected 2-regular graph G that is not an odd cycle, it is immediate that $C_2(G)$ is a Kempe class.)

Theorem (Theorems 3.2 and 3.14 combined)**.** *Let $\Delta \geq 1$. Let G be a connected graph with maximum degree Δ. If G is not $K_{\Delta+1}$ or if n is odd, C_n, then $C_\Delta(G)$ is a Kempe class unless $\Delta = 3$ and G is the triangular prism.*

The case $k = 3$ is considered in Section 3.2 and is based on published joint work with Matthew Johnson and Daniël Paulusma [52]. Its journal version has been submitted for publication [53]. The remaining case is addressed in Section 3.3 and is based on

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joint work with Marthe Bonamy, Nicolas Bousquet and Matthew Johnson, and has been submitted for publication [8].

In Chapter 4 we consider the computational complexity of a vertex partitioning problem restricted to special classes of graphs. Chudnovsky described in [29, 30] a complete characterization of bull-free graphs (a graph is bull-free if it contains no subgraph isomorphic to the bull graph – a graph on five vertices x_1, x_2, x_3, x_4, x_5 with edge set $\{x_1x_2, x_1x_3, x_2x_3, x_2x_4, x_3x_5\}$). Motivated by finding an algorithmic version of this structure theorem, the complexity of recognizing the class $\mathcal{T}_1$ described in [30] was posed as an open question by Thomassé, Trotignon and Vušković [111]. A graph $G \in \mathcal{T}_1$ if there exists a partition of $V(G) = V_1 \cup V_2$ such that V_1 induces a disjoint union of complete graphs and V_2 induces an independent set together with some prescribed adjacencies between V_1 and V_2 – for a full description of $\mathcal{T}_1$ the reader is referred to [30]. In an attempt to answer the question, we shall consider the following broader class of graphs. Call a graph $G = (V, E)$ *partitioniable* if there is a partition $\{V_1, V_2\}$ of V such that $G[V_1]$ is triangle-free and $G[V_2]$ is a disjoint union of complete graphs. Given that no adjacencies between V_1 and V_2 are prescribed in the definition of partitionable graphs, the class of partitionable bull-free graphs is a superclass of the class T_1. We prove the following theorem.

Theorem (Theorems 4.3 and 4.4 combined)**.** *Recognizing partitionable graphs is polynomial-time solvable when restricted to the class of cographs and* NP*-complete when restricted to the following classes:*

(1) planar graphs,

(2) K_4-free graphs,

(3) bull-free graphs,

(4) perfect graphs.

Chapter 4 is based on published joint work with Faisal N. Abu-Khzam and Haiko Müller [1].

In Chapter 5 we consider graph theoretic versions of a famous result in extremal combinatorics due to Erdős, Ko and Rado [47]. The extremal characterization was

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given by Hilton and Milner [70].

EKR Theorem (Erdős, Ko, Rado [47]; Hilton, Milner [70]) *Let n and r be positive integers, $n \geq r$, let S be a set of size n and let $\mathcal{A}$ be a family of subsets of S each of size r that are pairwise intersecting. If $n \geq 2r$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

Moreover, if $n > 2r$ the upper bound is attained only if the sets in $\mathcal{A}$ contain a fixed element of S.

There exists numerous proofs of the EKR theorem (see [60, 83] for example) as well as various analogues (see [61, 62, 72, 85] for example). The graph analogue introduced by Holroyd, Spencer and Talbot [72] is defined as follows: given a graph G and an integer $r \geq 1$, let $\mathcal{I}^{(r)}(G)$ denote the family of independent sets of size r of G. For a vertex v of G, let $\mathcal{I}_v^{(r)}(G)$ denote the family of independent sets of size r that contain v. This family is called an r-star. Then G is said to be r-EKR if no pairwise intersecting subfamily of $\mathcal{I}^{(r)}(G)$ is bigger than the largest r-star, and if every maximum size pairwise intersecting subfamily of $\mathcal{I}^{(r)}(G)$ is an r-star, then G is said to be strictly r-EKR. Let $\mu(G)$ denote the minimum size of a maximal independent set of G. Holroyd and Talbot conjectured that if $2r \leq \mu(G)$, then G is r-EKR and strictly r-EKR if $2r < \mu(G)$.

We consider two large subfamilies of trees: the class of *depth-two claws* and the class of *elongated claws with a short limb.*

Theorem (Theorems 5.8 and 5.9 combined)**.** *Let $r, n \geq 1$. Let G be a depth-two claw, and let H be an elongated claw with n leaves and a short limb. Then the following holds:*

- *G is strictly r-EKR if $2r \leq \mu(G) + 1$.*
- *H is r-EKR if $2r \leq n$.*

Chapter 5 is based on joint work with Matthew Johnson and Daniel Thomas and has been submitted for publication [55].

In addition to the above-mentioned results the thesis is supplemented with the following work. Chapter 3 investigates the implications of Theorem 3.14 on the

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ergodicity of the WSK algorithm in statistical mechanics while Chapters 2 and 5 contain further results as well as pose some conjectures and problems. All of these can be found in their respective journal versions. Unpublished work obtained by the author can also be found in Sections 2.5 and 3.4.

For the convenience of the reader the definitions in this section will be repeated in later chapters.

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Chapter 2

A Reconfigurations Analogue of Brooks' Theorem and its Consequences

2.1 Introduction

Recall the celebrated theorem of Brooks [24] which states that a connected graph G has a Δ-colouring unless G is the complete graph on $\Delta + 1$ vertices or a cycle with an odd number n of vertices. Our goal is to translate Brooks' Theorem to the setting of *reconfiguration graphs*.

Given a search problem (a computational problem that asks for a solution to be found), one can define a corresponding reconfiguration graph as follows: vertices correspond to solutions and edges join solutions that are, in some sense, "close" to one another. As this definition suggests, for a given search problem there might be more than one way to define an edge relation of the reconfiguration graph. Reconfiguration graphs have not only been studied for colouring, but also for many other problems including boolean satisfiability [65, 89, 107], clique and vertex cover [75], independent set [12, 14, 81], list edge colouring [76, 78], $L(2,1)$-labeling [77], shortest path [10, 11], and subset sum [74]; see also a recent survey of van den Heuvel [67] or the PhD thesis of Mouawad [101] for an excellent exposition. Typical questions are: is the reconfiguration graph connected; if so what is its diameter; if not what is the

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diameter of its (connected) components; and how difficult is it to decide whether there is a path between a pair of given solutions? Recent work has included looking at finding the shortest path in the reconfiguration graph between given solutions [80], and studying the fixed-parameter-tractability of this problem [15, 79, 102, 103].

For the colouring problem, two closely related definitions of the reconfiguration graph can be found in the literature. In particular we have reconfiguration steps defined as *Kempe changes* or as *recolourings*. A Kempe change is the exchange of colours of a component of the graph induced by two colours and a recolouring is a Kempe change of a component containing *exactly* one vertex. The reconfiguration graph having as reconfiguration steps Kempe changes can be traced back to 1879 in Kempe's well known fallacious proof of the Four Colour Theorem [84]. The Kempe change method has since proved to be a powerful tool both in theory (see, e.g., [57, 94, 97]) and in practice (see, e.g., [104, 116, 117]) and we refer the reader to Chapter 3 for a detailed survey and contribution on the topic. Reconfiguration graphs having as reconfiguration steps recolourings have received much more attention (see Section 2.1.1) and in the remainder of this chapter we are concerned with the latter definition. With slight abuse of notation, we refer to the *k-colouring reconfiguration graph* of G, denoted $R_k(G)$, as the graph with vertex set all possible k-colourings of G, and two k-colourings γ_1 and γ_2 are joined by an edge if γ_1 is obtained from γ_2 by a recolouring step; in other words, if for some vertex $u \in V$, $\gamma_1(u) \neq \gamma_2(u)$, and, for all $v \in V \setminus \{u\}$, $\gamma_1(v) = \gamma_2(v)$; that is, if γ_1 and γ_2 *disagree* on exactly one vertex.

As mentioned, besides determining a bound on the diameter of the reconfiguration graph or of its components, another common aim in this area is to decide whether or not there is a path between a given pair of colourings α and β in a reconfiguration graph. This leads to the following decision problem (where k denotes a fixed integer, that is, k is not part of the input):

k-COLOUR PATH

Instance: A graph $G = (V, E)$ and two k-colourings α and β.

Question: Is there a path in $R_k(G)$ between α and β?

Note that an equivalent formulation of this problem is whether there exists a se-

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quence of colourings $\gamma_0, \gamma_1, \ldots, \gamma_t$ with $\alpha = \gamma_0$, $\beta = \gamma_t$ such that adjacent colourings disagree on a single vertex. We call this a *recolouring* sequence. If, for $1 \leq i \leq t$, v_i is the vertex on which γ_i and γ_{i-1} disagree, then we can think of β as being obtained from α by recolouring the vertices $v_1, \ldots, v_t$ in order. Therefore, rather than explicitly considering the reconfiguration graph, one could seek to find a recolouring sequence of G; that is, to describe a sequence of vertices and to say which colour each vertex should be recoloured (while avoiding that two adjacent vertices are coloured alike).

2.1.1 Existing Results

The study of reconfiguration graphs of colourings was initiated by Cereceda, van den Heuvel and Johnson [26, 27] who proved some initial results on the connectivity of reconfiguration graphs. k-Colour Path was shown to be solvable in time $O(n^2)$ for $k = 3$ by Cereceda, van den Heuvel and Johnson [28]; they also proved that the diameter of any component of the reconfiguration graph $R_3(G)$ of a 3-colourable graph G is $O(n^2)$. In contrast, Bonsma and Cereceda [13] proved that this problem is PSPACE-complete for $k = 4$ even for bipartite graphs (and for planar graphs for $4 \leq k \leq 6$), and examples of reconfiguration graphs with components of superpolynomial diameter were given in all these cases.

Bonamy et al. [9] showed that reconfiguration graphs of k-colourings of chordal graphs are connected with diameter $O(n^2)$ whenever k is more than the size of the largest clique (and they gave an infinite class of chordal graphs whose reconfiguration graphs have diameter $\Omega(n^2)$). The proof idea of the former result is by induction on the number of vertices: since a pair u, v of vertices of a chordal graph G whose identification also results in a chordal graph H can be found, it follows by induction that all k-colourings of H can be obtained from one another by a sequence of $O(n^2)$ recolouring steps. To complete the proof, a sequence of $O(n^2)$ recolouring steps from any k-colouring of G to a k-colouring that colours u and v alike is described.

Bonamy and Bousquet [7] generalized this result by showing that if k is at least two greater than the treewidth $tw(G)$, then, again, $R_k(G)$ is connected with diameter $O(n^2)$; note that if $k = tw(G) + 1$, then $R_k(G)$ might not be connected since, for

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example, G might be a complete graph on $tw(G)+1$ vertices and then $R_k(G)$ contains no edges. Their approach is as follows: say that an independent set S of vertices is *merged* into a single vertex z if the vertices in S are replaced by the vertex z and, for each vertex $x \in V \setminus S$, xz is an edge if and only if there exists a vertex $y \in S$ such that xy is an edge in G. They show that there exists a sequence of $O(n^2)$ recolouring steps from any k-colouring to a k-colouring satisfying the following property: there exists a family $S_1, \ldots, S_p$ of independent sets in G such that, for $i = 1, \ldots, p$, the vertices in S_i are coloured alike and merging each of these sets into a single vertex results in a complete graph. As $R_k(K_n)$ is straightforwardly shown to have diameter $O(n^2)$ provided that $n \leq k+1$ the result then follows with a few extra considerations.

Bousquet and Perarnau [20] considered sparse graphs. They proved that, for all $d \geq 0$, $k \geq d$ and $\epsilon > 0$, the reconfiguration graph $R_k(G)$ of every $(d+1)$-colourable graph G has a polynomial diameter provided that the maximum average degree of G is at most $d - \epsilon$ (the maximum average degree of a graph G is defined as $\frac{2|E(G)}{|V(G)|}$). The proof is by induction on the average degree of the graph: a sequence of $O(n^2)$ recolouring steps from any $(d+1)$-colouring of the graph to a d-colouring is described. As colour $d + 1$ is not used in the resulting colouring, a subset $S \subset V$ is recoloured with colour $d+1$ with the property that the graph $G - S$ has average degree at most $d - 1 - \epsilon$. The proof then proceeds by applying the induction hypothesis to $G - S$. This approach is also used in the proofs of Theorem 1.1 in [7] and Theorem 2.2 of this chapter. We will mention other related results later.

2.1.2 Our Results

We study reconfigurations of colourings for graphs of bounded maximum degree. Our first result is an analogue of Brooks' theorem for reconfiguration graphs, that is, we answer the question: given a k-colouring γ of G, $k \geq \Delta + 1$, is there a path from γ to a Δ-colouring in $R_k(G)$? (Note that, for any two integers k and k' with $k \geq k'$, every k'-colouring of G corresponds to a vertex of $R_k(G)$ since a k'-colouring is a k-colouring in which not all colours are used.)

In order to state our results we recall some definitions. A k-colouring γ of a graph is *frozen* if, for every vertex v, every colour except $\gamma(v)$ is used on the neighbours

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of v. Notice that a frozen colouring is an isolated vertex in $R_k(G)$. The length of a shortest path between colourings α and β in $R_k(G)$ is denoted by $d_k(\alpha, \beta)$. We state our result for connected graphs as disconnected graphs can be considered component-wise.

Theorem 2.1. *Let G be a connected graph on n vertices with maximum degree $\Delta \geq 1$, and let $k \geq \Delta + 1$. Let α be a k-colouring of G. If α is not frozen and G is not $K_{\Delta+1}$ or, if n is odd, C_n, then there exists a Δ-colouring γ of G such that $d_k(\alpha, \gamma)$ is $O(n^2)$.*

Note that α can only be frozen if $k = \Delta + 1$, and only if G is Δ-regular. Let us briefly note that such colourings do exist: for example a 3-colouring of C_6 in which each colour appears exactly twice on vertices at distance 3, or a 4-colouring of the cube in which diagonally opposite vertices are coloured alike. In fact, as we will see, the case $k = \Delta + 1$ is the only cause of difficulty in the proof of our first result, which can be found in Section 2.2.

Using Theorem 2.1 we can, with the aid of a result of Matamala [90] on partitioning graphs into two degenerate graphs, give a characterization of $R_{\Delta+1}(G)$ for $\Delta \geq 3$, which is our next result and is proved in Section 2.3.

Theorem 2.2. *Let G be a connected graph on n vertices with maximum degree $\Delta \geq 3$. Let α and β be $(\Delta + 1)$-colourings of G. If α and β are not frozen colourings, then $d_{\Delta+1}(\alpha, \beta)$ is $O(n^2)$.*

Theorem 2.2 implies that $R_{\Delta+1}(G)$ contains a number of isolated vertices (representing frozen colourings) plus, possibly, one further component. We observe that the requirement that $\Delta \geq 3$ is necessary since, for example, $R_3(C_n)$, n odd, has more than one component consisting of at least two vertices [26, 27].

It is possible that the number of isolated vertices is zero; that is, there are no frozen $(\Delta+1)$-colourings. For example, suppose that G is a connected regular graph on $n \not\equiv 0 \mod (\Delta+1)$ vertices with maximum degree $\Delta \geq 3$, and let $V_1, V_2, \ldots, V_{\Delta+1}$ be the colour classes of a frozen $(\Delta + 1)$-colouring γ. Then, by definition, for all i, j, $i \neq j$, each $v \in V_i$ has a neighbour in V_j and cannot have more than one neighbour in V_j, as it has Δ neighbours in total. Hence, $|V_1| = \cdots = |V_{\Delta+1}|$ and

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thus $n \equiv 0 \mod (\Delta+1)$, a contradiction. We note that connected Δ-regular graphs on n vertices can always be found (unless n and Δ are both odd): for example, take n vertices arranged on a circle and join each to the nearest $\lfloor \Delta/2 \rfloor$ vertices on either side and also, if Δ is odd, to the diametrically opposite vertex.

It is also possible that there are only isolated vertices. Consider $R_4(K_4)$ for instance; and Brooks' Theorem tells us that complete graphs are the only graphs for which $R_{\Delta+1}(G)$ is edgeless, since other graphs have colourings in which only Δ colours are used and by recolouring any vertex with the unused colour we find a neighbouring colouring.

Though we will not directly use it, let us discuss a result that implies some cases of Theorem 2.2. First a definition: the Grundy number of a graph is the maximum number of colours needed if the vertices of the graph are coloured greedily. In [7], it was shown that, for any graph G on n vertices and any positive integer k, if k is greater than the Grundy number of G, then there is a path of length $O(n^2)$ between any pair of k-colourings of G. As it is well-known that the Grundy number of G is *at most* $\Delta+1$, this implies Theorem 2.2 except in the case that the Grundy number of G is *exactly* $\Delta + 1$.

GRUNDY NUMBER

Instance: A graph $G = (V, E)$.

Question: Does G have Grundy number at most Δ?

The decision problem GRUNDY NUMBER is coNP-complete even if we restrict our attention to bipartite graphs [67] (notice here that Δ is not fixed). In other words, the class of graphs with Grundy number $\Delta + 1$ is coNP-complete to recognize. Therefore the class of graphs with Grundy number $\Delta + 1$ is unlikely to admit a finite list of forbidden induced subgraphs (if the list were of finite size, its members could be recognized in constant time).

2.1.3 Two Characterization Results

Theorem 2.2 enables us to complete both a structural characterization and an algorithmic characterization for reconfiguration graphs of colourings of graphs with bounded maximum degree. In order to explain this we need to introduce some more

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terminology.

Throughout the chapter let n denote the number of vertices of a graph. We distinguish four types of classes of k-colourable graphs for our structural characterization. As we will see, these four types also roughly correspond to four types of complexity results. We say that a graph class $\mathcal{G}$ of k-colourable graphs is of

- **type 1** if, for all $G \in \mathcal{G}$, $R_k(G)$ is connected and has diameter $O(n^2)$;
- **type 2** if, for all $G \in \mathcal{G}$, each component of $R_k(G)$ has diameter $O(n^2)$ and $R_k(G)$ has at most one component that is not an isolated vertex;
- **type 3** if, for all $G \in \mathcal{G}$, each component of $R_k(G)$ has diameter $O(n^2)$;
- **type 4** if $\mathcal{G}$ contains an infinite family of graphs G such that $R_k(G)$ is disconnected and has at least one component with a superpolynomial diameter.

Note that every graph class of type 1 is of type 2 and that every graph class of type 2 is of type 3. At this point the reader may wonder whether there exists a class of graphs whose reconfiguration graph of k-colourings is connected, but does not have an (at most) quadratic diameter. This is still an open problem (see, for example, [9]). The structural characterization presented in Theorem 2.3 below implies that if such a graph class exists, then it contains graphs whose maximum degree is unbounded.

For integers $k \geq 1$ and $\Delta \geq 0$, let $\mathcal{G}_k^\Delta$ be the class of connected k-colourable graphs with maximum degree Δ. Note that $\mathcal{G}_1^\Delta = \emptyset$ if $\Delta \geq 1$ and that $\mathcal{G}_k^i \subseteq \mathcal{G}_k^j$ for any two integers i and j with $i \leq j$.

We are now ready to formally state the consequences of our earlier results. Theorem 2.3 describes the connectivity and the diameter of the reconfiguration graph of a graph of bounded degree in terms of the four types defined above. Theorem 2.4 completely determines the computational complexity of k-Colour Path restricted to graphs of bounded degree. We obtain these two characterization results by combining Theorem 2.2 with a number of results from the literature.

Theorem 2.3. *Let $k \geq 1$ and $\Delta \geq 0$ be integers. Then:*

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(i) $\mathcal{G}_k^\Delta$ *is of type 1 if*

- $k = 1$ *and* $\Delta = 0$
- $k \geq 2$ *and* $\Delta \leq k - 2$.

(ii) $\mathcal{G}_k^\Delta$ *is of type 2 if*

- $k = 2$ *and* $\Delta \geq 1$
- $k \geq 4$ *and* $\Delta = k - 1$.

(iii) $\mathcal{G}_k^\Delta$ *is of type 3 if*

- $k = 3$ *and* $\Delta \geq 2$.

(iv) $\mathcal{G}_k^\Delta$ *contains a subclass of type 4 if*

- $k \geq 4$ *and* $\Delta \geq k$.

Proof. We prove each of the four statements separately.

(i) The case $k = 1$ and $\Delta = 0$ is trivial. The case $k \geq 2$ and $\Delta \leq k - 2$ has been shown by Dyer, Flaxman, Frieze and Vigoda [42]; see also [13, 26, 25] for a proof.

(ii) The case $k = 2$ and $\Delta \geq 1$ follows from the fact that $\mathcal{G}_2^\Delta$ consists of connected bipartite graphs. Hence, the corresponding reconfiguration graphs are either edgeless or isomorphic to a single edge (if the bipartite graph consists of a single vertex). The case $k \geq 4$ and $\Delta = k - 1$ follows from Theorem 2.2.

(iii) This case has been proven by Cereceda, van den Heuvel and Johnson [28].

(iv) Let $k \geq 4$ and $\Delta \geq k$. Bonsma and Cereceda [13] constructed an infinite family of k-colourable graphs whose reconfiguration graphs have components of superpolynomial diameter. It can be observed that these graphs belong to $\mathcal{G}_k^k$, and hence, to $\mathcal{G}_k^\Delta$ for all $\Delta \geq k$.

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Theorem 2.4. *Let $k \geq 1$ and $\Delta \geq 0$ be integers. Then k-*Colour Path *restricted to $\mathcal{G}_k^\Delta$ is*

(i) solvable in $O(n)$ time if

- $k \leq 2$
- $k \geq 3$ *and* $\Delta \leq k-2$;

(ii) solvable in $O(n^2)$ time if

(*a*) $k \geq 3$ *and* $\Delta = k-1$;

(*b*) $k = 3$ *and* $\Delta \geq 3$;

(iii) PSPACE*-complete if*

- $k \geq 4$ *and* $\Delta \geq k$.

Proof. We prove each of the four statements separately.

(i) This case follows from Theorem 2.3 (i) (the answer is always yes) unless $k = 2$ and $\Delta \geq k-1 = 1$. Recall from the proof of Theorem 2.3 (ii) that in the latter case the reconfiguration graph is either edgeless or isomorphic to an edge. The answer is always no in the first case and yes in the second case.

(ii)(a) If $k = 3$ and so $\Delta = 2$, then G is either a path or a cycle. We know k-Colour Path always has the answer yes for paths [25], and can be decided for cycles by a single traversal of the edges [28]. Now let $k \geq 4$. By Theorem 2.3 (ii), it is necessary in this case only to check for each vertex v in the input graph G, for each of the two given k-colourings α and β, whether v and its neighbours use every colour in $\{1, 2, \ldots, \Delta+1\}$. If they do not, neither colouring is frozen, so there is a path between them.

(ii)(b) This follows from [28] for the superclass consisting of all 3-colourable graphs.

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(iii) This follows from the aforementioned result of Bonsma and Cereceda [13] as from their proof it can be seen that the problem is PSPACE-complete for $\mathcal{G}_k^k$, and thus for $\mathcal{G}_k^\Delta$ for all $\Delta \geq k$.

■

2.1.4 Further Work and Open Problems

We already mentioned the open problem on the existence of a class of graphs whose reconfiguration graph of k-colourings is connected, but does not have an (at most) quadratic diameter. We recall another open problem from the literature which is on degenerate graphs and on which we can report some partial progress due to our new results. A graph G is *k-degenerate* if every induced subgraph of G has a vertex with degree at most k. Note that any graph is Δ-degenerate. Cereceda [25] made the following conjecture.

Conjecture 2.5. *For any pair of integers d, k with $k \geq d+2$, the reconfiguration graph $R_k(G)$ of a d-degenerate graph G has diameter $O(n^2)$.*

It turns out that proving (or disproving) this conjecture is a very challenging problem even for $d = 2$ and $k = 4$. Using Theorem 2.2 we can solve one more case, as shown in the next theorem which summarizes our current knowledge.

Theorem 2.6. *Let $d \geq 0$ and $k \geq d+2$, and let G be a d-degenerate connected graph. Then $R_k(G)$ has diameter $O(n^2)$ if*

(i) $d = 0$

(ii) $d = 1$

(iii) $d = \Delta - 1$

(iv) $d \geq \Delta$.

Proof. We prove each of the four statements separately.

(i) This case is trivial.

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(ii) Cereceda [25] proved that for any two integers d and k with $k \geq 2d+1$, the reconfiguration graph $R_k(G)$ of any d-degenerate graph G has diameter $O(n^2)$. Taking $d = 1$ proves the case. As an aside, Bousquet and Perarnau [20] proved that for any two integers d and k with $k \geq 2d+2$, the reconfiguration graph $R_k(G)$ of any d-degenerate graph G has diameter $O(n)$.

(iii) If $k = d+2 = \Delta+1$ then we can apply Theorem 2.2 after observing that a $(\Delta-1)$-degenerate graph has a vertex with at most $\Delta-1$ neighbours, so no k-colouring α is frozen. If $k \geq d+3 = \Delta+2$ then we apply Theorem 2.3 (i).

(iv) This case follows from Theorem 2.3 (i).

■

Another direction for future work is to consider the problem of *finding* a path or a shortest path in the reconfiguration graph $R_k(G)$ between two given k-colourings α and β of a graph G of maximum degree Δ. For $k \geq 4$ and $\Delta \geq k$ this problem is PSPACE-hard due to Theorem 2.4 (iv). However, for $1 \leq k \leq 3$ or $0 \leq \Delta \leq k-1$, this problem is not solved in statements (i)–(iii) of Theorem 2.4, which correspond to exactly those cases for which k-Colour Path is polynomial-time solvable, but which only provide a yes-answer or no-answer in polynomial time. Note that the maximum degree of $R_k(G)$ could be equal to $(k-1)n$. This bound, together with an $O(n^2)$ bound on its diameter, only imply an $(kn)^{O(n^2)}$ bound on the running time of a Breadth-First Search starting in one of the colourings α, β.

Let us discuss what is known for $1 \leq k \leq 3$ or $0 \leq \Delta \leq k-1$. First of all, the problem is trivial to solve if $k \leq 2$. For $k = 3$, Johnson et al. [79] proved that it is possible in $O(n+m)$ time to find even a shortest path between two given k-colourings in the reconfiguration graph $R_3(G)$ of any 3-colourable graph G with n vertices and m edges. The case $0 \leq \Delta \leq k-2$ has been shown to be solvable in $O(n^2)$ time by Cereceda [25]. This leaves us with the case $\Delta = k-1$ and $k \geq 4$, or equivalently, $\Delta \geq 3$ and $k = \Delta+1$. For this case we have the following result, the proof of which can be found in Section 2.4.

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Theorem 2.7. *Let G be a connected graph on n vertices with maximum degree $\Delta \geq 3$. Let $k = \Delta + 1$. If G is not regular, then it is possible to find in $O(n^2)$ time a path between any two given k-colourings α and β in $R_k(G)$.*

Hence, the only remaining case, which we leave as an open problem, is when $\Delta \geq 3$, $k = \Delta + 1$ and G is Δ-regular. We believe that solving this case is non-trivial, because the straightforward approach of modifying the structural proof of Theorem 2.2 does not work. As explained in Section 2.4, such an approach would require us to find a maximum independent set for graphs of bounded maximum degree in polynomial time. However, this problem is NP-complete even for cubic graphs [63].

2.2 The Proof of Theorem 2.1

In order to prove Theorem 2.1, we need a number of lemmas that are mostly concerned with $(\Delta+1)$-colouring. We define a number of terms we will use to describe vertices of G with respect to some $(\Delta+1)$-colouring. A vertex v is *locked* if Δ distinct colours appear on its neighbours. A vertex that is not locked is *free*. Clearly a vertex can be recoloured only if it is free. If v is locked and then one of its neighbour is recoloured and v becomes free, we say that v is *unlocked*. A vertex v is *superfree* if there is a colour $c \neq \Delta+1$ such that neither v nor any of its neighbours is coloured c. A vertex can be recoloured with a colour other than $\Delta + 1$ if it is superfree. Note there are $\Delta-1$ distinct colours that must appear on the Δ neighbours of v if it is not superfree. We say that G is in $(\Delta+1)$-*reduced form* if for every vertex v coloured with $\Delta+1$, v and each of its neighbours are locked. This implies that the distance between any pair of vertices coloured $(\Delta+1)$ is at least 3 as no locked vertex can have two neighbours coloured $(\Delta+1)$.

The key to proving Theorem 2.1 will be to show that from a $(\Delta+1)$-colouring one can recolour some of the vertices to arrive at a colouring in which colour $\Delta+1$ appears on fewer vertices. We begin by considering the case where the colour $\Delta+1$ appears on only one vertex. The proof of the following lemma is inspired by a proof of Brooks' Theorem [92], but also uses some new arguments.

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Lemma 2.8. *Let $G = (V, E)$ be a connected graph on n vertices with maximum degree $\Delta \geq 3$, and let α be a $(\Delta + 1)$-colouring of G with exactly one vertex v coloured $\Delta + 1$. If G does not contain $K_{\Delta+1}$ as a subgraph, then there exists a Δ-colouring γ of G such that $d_k(\alpha, \gamma)$ is $O(n)$.*

Proof. We can assume that G is in $(\Delta + 1)$-reduced form since if v is not locked, then we can immediately recolour it; if a neighbour of v is not locked, then it can be recoloured and this will unlock v and allow us to recolour it.

Let us fix a labelling of the neighbours of v: let x_i be the neighbour such that $\alpha(x_i) = i$, $1 \leq i \leq \Delta$. Our aim is to find a recolouring sequence that unlocks v. There is one recolouring sequence that we will use several times. Suppose that C is a connected component of a subgraph of G induced by two colours i and j, $\Delta + 1 \notin \{i, j\}$, and no vertex coloured j in C is adjacent to v. First the vertices coloured j are recoloured with $\Delta + 1$. Then the vertices coloured i are recoloured j, and finally the vertices initially coloured j are recoloured i. It is clear that all colourings are proper and the overall effect is to *swap* the colours i and j on C.

We say that any colouring γ where G is in $(\Delta + 1)$-reduced form, only v is coloured $\Delta + 1$ and $\gamma(x_i) = i$, $1 \leq i \leq \Delta$, is *good*. For any good colouring γ, let G^{γ}_{ij} be the maximal connected component containing x_i of the subgraph of G induced by the vertices coloured i and j by γ.

We make some claims about good colourings. When we claim that v can be unlocked, it is implicit that colour $\Delta + 1$ is not used on any other vertex in the graph so that unlocking v allows us to reach a colouring where $\Delta + 1$ is not used.

Claim 1: If γ is good and $x_j \notin G^{\gamma}_{ij}$, then v can be unlocked.

If $x_j \notin G^{\gamma}_{ij}$, then the only vertex adjacent to v in G^{γ}_{ij} is x_i. Thus the colours i and j can be swapped on G^{γ}_{ij}. Then v has two neighbours with colour j and is unlocked.

Claim 2: If γ is good and G^{γ}_{ij} is not a path from x_i to x_j, then v can be unlocked.

By Claim 1, we can assume that x_i and x_j are in G^{γ}_{ij}. They must have degree 1 in G_{ij} since, as G is in $(\Delta + 1)$-reduced form, they are locked. Suppose that G^{γ}_{ij} is not a path and consider the shortest path in G^{γ}_{ij} from x_i to x_j, and the vertex w nearest to x_i on the path that has degree more than 2. Then w has at least three

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neighbours coloured alike in G and is superfree and can be recoloured with a colour other than i, j or $\Delta+1$. Call this new colouring γ' and note that, by the choice of w, $G_{ij}^{\gamma'}$ does not contain x_j. Now Claim 1 implies Claim 2.

As G is $K_{\Delta+1}$-free, v and its neighbours are not a clique so we can assume that x_1 and x_2 are not adjacent. Let u be the unique neighbour of x_1 coloured 2. For a good colouring γ, note that u is in G_{12}^{γ}, and let H_{23}^{γ} be the component of the subgraph of G induced by the vertices with colour 2 and 3 that contains u.

Claim 3: If γ is good and u has more than one neighbour in H_{23}^{γ}, then v can be unlocked.

If G_{12}^{γ} is not a path, then use Claim 2. Otherwise u has two neighbours coloured 1; if u has two neighbours in H_{23}^{γ}, then it also has two neighbours coloured 3 and is superfree. Recolour it and apply Claim 1.

Claim 4: If γ is good and H_{23}^{γ} is a path, then v can be unlocked.

By Claim 2 we can assume G_{23}^{γ} is a path. If $H_{23}^{\gamma}=G_{23}^{\gamma}$, then we can use Claim 3. So we assume $H_{23}^{\gamma}\neq G_{23}^{\gamma}$ and so $x_2, x_3 \notin H_{23}$ and H_{23} contains no neighbour of v. Let γ' be the colouring obtained by swapping the colours 2 and 3 on H_{23}^{γ}.

By Claim 3, u is an endvertex of H_{23}^{γ}. Let the other endvertex be w. (If $w=u$, then u has no neighbour coloured 3 and can be recoloured. Then use Claim 2.)

If $G_{12}^{\gamma'}$ is not a path from x_1 to x_2, we use Claim 2. If it is such a path, then let the unique neighbour of x_1 in $G_{12}^{\gamma'}$ be y and clearly $y\in H_{23}^{\gamma}$. From x_2 traverse $G_{12}^{\gamma'}$ until the last vertex z that is also in G_{12}^{γ} is reached. Let t be the next vertex along from z towards x_1 in $G_{12}^{\gamma'}$. Clearly t is also in H_{23}^{γ}. In fact, we can assume that $w=y=t$ since if y or t has degree 2 in H_{23} as well as in $G_{12}^{\gamma'}$ it has two neighbours coloured 1 and two neighbours coloured 3 in γ' and is superfree. It can be recoloured and then Claim 2 is used.

So x_1wz is coloured 131 in γ so is in G_{13}^{γ}. Then z is in both G_{13}^{γ} and G_{12}^{γ} so is superfree and can be recoloured so that Claim 2 can be used. This completes the proof of Claim 4.

To complete the proof: we know that the initial colouring α is good. If none of the four claims can be used, then consider H_{23}^{α}. We know that u has degree 1 in H_{23}, but H_{23} is not a path. So traversing edges away from u in H_{23}^{α}, let s be the

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first vertex reached with degree 3. Then s is superfree and can be recoloured so that H_{23} becomes a path, and then Claim 4 can be used. ■

In Lemma 2.10, we shall see how the number of vertices coloured $\Delta + 1$ can be reduced when more than one such vertex is present. First we need some definitions and a lemma. Let P be a path:

- P is *nearly* $(\Delta + 1)$*-locked* if its endvertices are locked and coloured $\Delta + 1$;
- P is $(\Delta + 1)$*-locked* if it is nearly $(\Delta + 1)$-locked and every vertex on the path is locked.

Lemma 2.9. *Let G be a graph in $(\Delta + 1)$-reduced form. If G has a $(\Delta + 1)$-locked path P, then each endvertex of P is an endvertex of an $(\Delta + 1)$-locked path of length 3.*

Proof. As we noted, by the definition of $(\Delta + 1)$-reduced form, a path between two vertices coloured $\Delta + 1$ has length at least 3. Let u be one endvertex of P and let Q be the shortest $(\Delta + 1)$-locked path that ends at u (so Q is induced). Let v be the vertex on Q at distance 2 from u. Then, as v is locked and not a neighbour of u, it has a neighbour w coloured $\Delta + 1$ that is not u and the path from u to w has length 3.

■

A path is *nice* if it is a nearly $(\Delta + 1)$-locked path, it contains free vertices and the endvertices and their neighbours are the only locked vertices. Notice that a nice path is not necessarily induced and, in particular, may contain a $(\Delta + 1)$-locked subpath. Notice that the definition implies that a nice path has at least five vertices.

Lemma 2.10. *Let G be a connected graph on n vertices with maximum degree $\Delta \geq 3$, let α be a $(\Delta + 1)$-colouring of G, and suppose that G is in $(\Delta + 1)$-reduced form. If G has at least two $(\Delta + 1)$-locked vertices and is not frozen, then there exists a $(\Delta + 1)$-colouring γ of G, such that $d_{\Delta+1}(\alpha, \gamma) = O(n)$ and fewer vertices are coloured $\Delta + 1$ with γ than with α.*

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Proof. We consider a number of cases.

Case 1: There exists a free vertex u adjacent to a $(\Delta+1)$-locked path P.

Let b be the vertex on the path adjacent to u. As b is locked it has a neighbour a coloured $\Delta+1$. Let c be a neighbour of b on P other than a. As c is locked it has a neighbour d coloured $\Delta+1$.

Since G is in $(\Delta+1)$-reduced form, u is not adjacent to a or d, but might be adjacent to c. In each case, it is routine to verify that by recolouring u to $\Delta+1$, b and c can both be recoloured unlocking a and d and allowing them to be recoloured. Thus the number of vertices coloured $\Delta+1$ is reduced.

Case 2: G has a nice path.

Let P be a shortest nice path. Let the endpoints be v and w with neighbours s and t on P respectively. If s and t are adjacent, then the path $vstw$ is $(\Delta+1)$-locked and has a free vertex adjacent to s so use Case 1. Thus assume that P is induced since the presence of any other edge would imply either a shorter nice path could be found or that the graph was not in $(\Delta+1)$-reduced form.

We use induction on the number ℓ of free vertices in P to show that there is a sequence of recolourings that lead to a colouring that has fewer vertices coloured $\Delta+1$.

If $\ell=1$, let u be the free vertex in P. Recolour u to $\Delta+1$. Now s and t have two neighbours coloured $\Delta+1$ and can be recoloured. Then v and w are unlocked and can both be recoloured, and this leaves one vertex on P coloured $\Delta+1$ rather than two.

Suppose that $\ell=2$. Let $P=vsu_1u_2tw$ where u_1 and u_2 are free vertices. First suppose that u_1 or u_2, say u_1, is superfree: recolouring u_1 to a colour $c \neq \Delta+1$ unlocks s; recoloured s unlocks v which, in turn, allows us to recolour it, and the number of vertices coloured $\Delta+1$ has been reduced as required. Similarly if u_1 is not superfree, but a neighbour x is, then x can be recoloured to a colour $c \neq \Delta+1$ and if xs is an edge, then s is unlocked and so v can be recoloured, and otherwise u is now superfree, the colours in the neighbourhood of s are unchanged, and the preceding argument can be applied.

Thus henceforth we can assume that u_1, u_2 and their neighbours are not superfree

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which implies that they have degree Δ.

Subcase 2.1: *u_1 and u_2 do not share a neighbour.* Let x_1 and x_2 be neighbours of u_1 and u_2 not in P. Clearly $x_1 \neq x_2$ and u_1x_2 and u_2x_1 are not edges.

Subcase 2.1.1: *x_1 is locked.* We know x_1 has a $(\Delta+1)$-locked neighbour, and this must be v (if it is some other vertex z, then vsu_1x_1z is a nice path that is shorter than P).

Suppose x_1s is not an edge. Recolour u_1 to $\Delta+1$. This unlocks x_1 which can be recoloured with $\alpha(u_1)$ which, in turn, unlocks v and allows us to recolour it with $\alpha(x_1)$. If u_1 is free, it can be recoloured and the number of vertices coloured $\Delta+1$ is reduced and we are done. If u_1 is locked, then note that s has been unlocked (as it no longer has a neighbour coloured $\alpha(u_1)$). Thus we can recolour s and then recolour u_1 with $\alpha(s)$ and again we have removed one instance of the colour $\Delta+1$.

Suppose instead that x_1s is an edge. Notice that $\alpha(s)$, $\alpha(u_1)$ and $\alpha(x_1)$ are distinct as the three vertices form a triangle. Recolour u_1 with $\Delta+1$ and then s with $\alpha(u_1)$. Now v is unlocked and can be recoloured with $\alpha(s)$. If u_1 is free, then recolour it and we are done. Otherwise this sequence of recolourings leaves u_1 locked (with $\alpha(u_1)$ and $\alpha(x_1)$ as the colours on s and x_1 respectively). So, from α, we do the following instead: again start by recolouring u_1 with $\Delta+1$, but then recolour x_1 with $\alpha(u_1)$ to unlock v. Now that $\alpha(x_1)$ is not used on a neighbour of u_1, u_1 is free and can be recoloured.

Subcase 2.1.2: *x_1 is free.* If x_2 is locked, we can, by symmetry, use the previous subcase, so we can assume that both x_1 and x_2 are free. Recolour u_2 to $\Delta+1$. Then t is unlocked and can be recoloured which, in turn, unlocks w allowing us to recolour it too. If u_2 is free, we recolour it and are done. If u_1 is free, we recolour it and unlock u_2 and, again, recolour it.

If u_1 and u_2 are both locked, observe that x_1 is still free as it has no neighbour coloured $\Delta+1$ since u_2x_1 is not an edge. Recolour x_1 to $\Delta+1$, and then recolour u_1 to $\alpha(x_1)$. Note that now s has no neighbour coloured $\alpha(u_1)$ and is free and can be recoloured. Thus v is unlocked and can also be recoloured. By recolouring u_1, we also unlock u_2, so we recolour it and are done.

Subcase 2.2: *u_1 and u_2 share a neighbour.* Let x_1 be a neighbour of u_1 and u_2.

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Since P is induced, x_1 is not in P. If x_1 is locked, then let its neighbour coloured $\Delta+1$ be y. Then vsu_1x_1y is a shorter nice path unless $y=v$. By an analogous argument we need $y=w$. This contradiction tells us that x_1 must be free.

If x_1 is joined to both s and t, then vsx_1tw is a shorter nice path. So, without loss of generality, assume that x_1t is not an edge. Thus as u_2 has a neighbour that is not adjacent to x_1, x_1 has a neighbour x_3 that is not adjacent to u_2 since both have degree Δ.

Subcase 2.2.1: $x_3 = s$. Recolour u_1 with $\Delta+1$ and then s with $\alpha(u_1)$. Now v is unlocked and can be recoloured with $\alpha(s)$. If u_1 is free, then recolour it and we are done. If u_2 or x_1 is still free, then recolour one of them to unlock u_1, which can, in turn, be recoloured and we are done. Otherwise this sequence of recolourings leaves u_1, u_2 and x_1 locked so x_1 is the only neighbour of u_2 coloured $\alpha(x_1)$. So, from α, we do the following instead: recolour x_1 with $\Delta+1$ to unlock s and then v. If x_1 can be recoloured, then we do so and we are done. Otherwise notice that $\alpha(x_1)$ is not used on a neighbour of u_2. It is thus free and can be recoloured to unlock x_1 which can then be recoloured.

Subcase 2.2.2: $x_3 \neq s$*, and* x_3 *is free.* First, suppose x_3s is an edge. Recolour u_2 to $\Delta+1$, t to $\alpha(u_2)$ and w to $\alpha(t)$. If either u_2 or one of its neighbours is now free, u_2 can be recoloured and we are done. Otherwise u_1, u_2 and x_1 are all locked, but x_3 is still free since it has no neighbour coloured $\Delta+1$. Recolour x_3 to $\Delta+1$ to unlock x_1; then recolour x_1 to unlock and recolour u_2. As x_3s is an edge, s has two neighbours coloured $\Delta+1$. Thus we recolour s to unlock v.

If x_3t is an edge we can use a similar argument. So suppose x_3s and x_3t are not edges. Recolour u_2 to $\Delta+1$, to unlock and recolour first t and then w. It is possible to recolour u_2 unless it and all its neighbours are locked. This implies that u_1, x_1 and u_2 are locked. We consider two subcases.

Subcase 2.2.2.1: u_1x_3 *is not an edge.* We recolour x_3 to $\Delta+1$ to unlock and recolour x_1 and then u_2. Notice that u_1 is now free since it has no neighbour coloured $\Delta+1$. Recoloured u_1 unlocks s, so we recolour it, which in turn unlocks v. Observe that x_1 now has two neighbours u_1 and x_3 with colour $\Delta+1$ so is free. If u_1 or u_3 is free, we can recolour at least one of them directly and we are done.

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Otherwise, we recolour x_1 so that x_3 and u_1 can now be recoloured.

Subcase 2.2.2.2: u_1x_3 *is an edge.* Recolour u_3 to $\Delta+1$, then recolour u_1, s and v. Observe that x_1 now has two neighbours u_2 and u_3 with colour $\Delta+1$. If u_2 or u_3 are free, we are done. Otherwise, recolour x_1, then recolour u_2 and x_3, and we are done.

Subcase 2.2.3: $x_3 \neq s$*, and* x_3 *is locked.* Then x_3 has a $(\Delta+1)$-locked neighbour y. If $y = v$, the path $H = vx_3x_1u_2tw$ is nice with two free vertices x_1 and u_2. Furthermore, u_1 is free and a neighbour of x_1 and u_2, in which case H satisfies the previous subcase unless x_3 and t are adjacent in which case use Subcase 2.1. A similar argument can be made if $y = w$ or $y \notin \{v, w\}$.

This completes the case $\ell = 2$. (We note that if we wished to use the proof to construct an algorithm, we would first check whether x_3 is superfree as in this case the proof can be simplified in many places.)

Now suppose that for all $i < \ell$, if there is a nice path containing i free vertices, the number of vertices coloured $\Delta+1$ can be reduced. Suppose that the shortest such path is $P = vsu_1u_2 \ldots u_\ell tw$ where $\ell \geq 3$. We recolour u_ℓ to $\Delta+1$, then t and then w. If u_ℓ or one of its neighbours is free, then u_ℓ can be recoloured and we are done. Otherwise, u_ℓ and $u_{\ell-1}$ are locked. Consider the path $P' = vsu_1 \ldots u_{\ell-2}u_{\ell-1}u_\ell$. By our inductive hypothesis, the number of colour $\Delta+1$ vertices in P' can be reduced. Case 2 is complete. (Let us remark that if $\ell = 2$, then we can construct P' in this way, but it will not be nice so it was necessary to consider that case separately.)

After Cases 1 and 2 we are left with:

Case 3: There does not exist a free vertex adjacent to a $(\Delta+1)$-locked path and G has no nice path.

As G contains more than one $(\Delta+1)$-locked vertex, it contains a nearly $(\Delta+1)$-locked path; let P be the shortest such path and let v and w be its endvertices. As G is in $(\Delta+1)$-reduced form, v, w and their neighbours are locked. If P contains no other vertices, it is $(\Delta+1)$-locked. Otherwise, since there are no nice paths, P contains another locked vertex u. Let y be the neighbour of u coloured $\Delta+1$. If y is on P, then we can assume, without loss of generality, that it is not between v and u. Then, whether or not y is on P, the subpath from v to u plus the edge uy is a

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shorter nearly $(\Delta + 1)$-locked path. This contradiction proves that G must contain a $(\Delta + 1)$-locked path.

As G is not frozen, it contains a free vertex. Let Q be the shortest path in G that joins a free vertex to a $(\Delta + 1)$-locked vertex. Let v be the $(\Delta + 1)$-locked endvertex. So v is an endpoint of a $(\Delta + 1)$-locked path R, and, by Lemma 2.9, we can assume that R has length 3.

Let u be the endvertex of Q that is free. By the minimality of Q, u is the only free vertex in Q. Let a be the neighbour of u in Q. As a is locked it has a $(\Delta+1)$-locked neighbour z. Thus we must have $z = v$ and $Q = vau$.

Let $R = wtsv$. Observe that us, ut, uv and uw cannot be edges as no locked path has a free neighbour. Thus the vertices of R and Q other than v are distinct. Consider the (not necessarily induced) path $M = wtsvau$. Notice also that at is not an edge, else the free vertex u is adjacent to the $(\Delta + 1)$-locked path $vatw$.

Suppose M is an induced path. Recolour u with $\Delta + 1$ to unlock and recolour a and then v. If u is not locked, then recolour and we are done. Else notice that the vertices v and s are free, and the vertices u, a, t, w are locked. Consequently, we have that M is a nice path, and by Case 2 we are done.

The only edge that might be present among the vertices of M is as so suppose this exists. Recolour u with $\Delta + 1$ to unlock and recolour first a and then v. If u or any of its neighbours are free, u can be recoloured and we are done. Otherwise note that recoloured v unlocks s. It follows that the path $H = uastw$ is nice, and we can use Case 2. This completes Case 3.

As each vertex is recoloured a constant number of times, the lemma follows. ■

We are now ready to prove Theorem 2.1, which we first restate.

Theorem 2.1. *Let G be a connected graph on n vertices with maximum degree $\Delta \geq 1$, and let $k \geq \Delta + 1$. Let α be a k-colouring of G. If α is not frozen and G is not $K_{\Delta+1}$ or, if n is odd, C_n, then there exists a Δ-colouring γ of G such that $d_k(\alpha, \gamma)$ is $O(n^2)$.*

Proof. If $k > \Delta + 1$, then, by Brooks' Theorem, a Δ-colouring γ exists in $R_k(G)$ unless G is complete or an odd cycle. We know that, in this case, $R_k(G)$ is connected

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and has diameter $O(n^2)$ so certainly $d_k(\alpha, \gamma)$ is $O(n^2)$.

Suppose that $k = \Delta + 1$ and G is in $(\Delta + 1)$-reduced form with α: if not, we try to recolour each vertex with colour $\Delta + 1$ either directly or by first recolouring one of its neighbours. Repeatedly applying Lemma 2.10 starting from α, we obtain a $(\Delta + 1)$-colouring γ' by $O(n^2)$ recolourings such that at most one vertex is coloured $(\Delta + 1)$ with γ'. Lemma 2.8 can now be applied to obtain a Δ-colouring γ from γ' by $O(n)$ recolourings. Hence, $d_{\Delta+1}(\alpha, \gamma) \leq O(n^2)$ as required. ∎

2.3 The Proof of Theorem 2.2

First we need the following result of Matamala [90]. We use $\omega(G)$ to denote the number of vertices in the largest clique in G.

Lemma 2.11 ([90]). *Let $G = (V, E)$ be a graph with maximum degree $\Delta \geq 3$ and $\omega(G) \leq \Delta$. Let p_1 and p_2 be non-negative integers such that $p_1 + p_2 = \Delta - 2$. Then there is a partition $\{S_1, S_2\}$ of V such that S_1 induces a maximum size p_1-degenerate graph in G and S_2 induces a p_2-degenerate graph.*

We also need the following two lemmas.

Lemma 2.12. *Let G be a connected $(\Delta - 1)$-degenerate graph on n vertices with maximum degree $\Delta \geq 3$, and let $k \geq \Delta + 1$. Let α be a k-colouring of G. Then there exists a Δ-colouring γ of G such that $d_k(\alpha, \gamma)$ is $O(n^2)$.*

Proof. The result follows immediately from Theorem 2.1 by observing that a $(\Delta - 1)$-degenerate graph has a vertex with at most $\Delta - 1$ neighbours, so α is not frozen and G is not $K_{\Delta+1}$ or C_n. ∎

Lemma 2.13. *Let $G = (V, E)$ be a graph on n vertices with maximum degree $\Delta \geq 1$. Let γ_1 and γ_2 be Δ-colourings of G. Then $d_{\Delta+1}(\gamma_1, \gamma_2)$ is $O(n^2)$.*

Proof. We use induction on Δ. If $\Delta \in \{1, 2\}$ the statement is trivially true. Let $\Delta \geq 3$. We observe that $\omega(G) \leq \Delta$ because G is Δ-colourable. Applying Lemma 2.11 with $p_1 = 0$ and $p_2 = \Delta - 2$, we obtain a partition $\{S_1, S_2\}$ of V such that S_1 is

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a maximum independent set and S_2 induces a $(\Delta - 2)$-degenerate graph that we denote by H.

From γ_1 and γ_2 recolour the vertices of S_1 with colour $\Delta + 1$ (the colour that is not used in either γ_1 or γ_2). This can be done by at most $2n$ recolourings. So now we can focus on the colourings restricted to S_2, and as long as we do not use the colour $\Delta + 1$ we do not need to worry about adjacencies with S_1. So let γ_1^H and γ_2^H be the colourings of H that are obtained by taking the restrictions of γ_1 and γ_2 to S_2, and if we can recolour from γ_1^H to γ_2^H by $O(n^2)$ recolourings without using colour $\Delta + 1$, we will be done. We note that

- γ_1^H and γ_2^H use only colours from $\{1, 2, \dots, \Delta\}$;
- each component of H has maximum degree at most $\Delta - 1$ (since every vertex in S_2 has at least one neighbour in S_1 by the maximality of S_1);
- each component of H is $(\Delta - 2)$-degenerate.

Thus we can apply Lemma 2.12 on each component of H to recolour each of γ_1^H and γ_2^H to a $(\Delta - 1)$-colouring using at most $O(n^2)$ recolourings. By the inductive hypothesis, there is a path of length $O(n^2)$ between these two $(\Delta - 1)$-colourings that includes only Δ-colourings so does not use colour $\Delta + 1$. Because at most $2n$ recolourings were needed to recolour γ_1 and γ_2 to γ_1^H and γ_2^H, the total number of recolourings is $O(n^2)$. This completes the proof of Lemma 2.13. ■

The lemma says that there is a path between any pair of Δ-colourings, but, because we are working with $R_{\Delta+1}(G)$, the intermediate colourings might use $\Delta + 1$ colours. We are now ready to prove Theorem 2.2, which we restate below.

Theorem 2.2. *Let G be a connected graph on n vertices with maximum degree $\Delta \geq 3$. Let α and β be $(\Delta + 1)$-colourings of G. If α and β are not frozen colourings, then $d_{\Delta+1}(\alpha, \beta)$ is $O(n^2)$.*

Proof. Theorem 2.1 implies that from each of α and β there is a path in $R_{\Delta+1}$ to a Δ-colouring; Lemma 2.13 implies that there is a path between these two Δ-colourings that completes the path from α to β. ■

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2.4 The Proof of Theorem 2.7

The *degeneracy* of a graph G is the least integer k such that G is k-degenerate.

We start with the following easy lemma, which is well known (see, for example, [90]). We give a short proof for completeness.

Lemma 2.14. *Let $\Delta \geq 1$. Every connected graph with maximum degree Δ that is not regular is $(\Delta - 1)$-degenerate.*

Proof. Let G be a smallest possible counterexample, so G has degeneracy and maximum degree equal to Δ and contains a vertex v with $\deg(v) < \Delta$. If $G - v$ has degeneracy Δ, then, by the minimality of G, we find that $G - v$ is Δ-regular. This means that in G every neighbour of v has more than Δ neighbours, which is not possible. Hence, $G - v$ has degeneracy $\Delta - 1$. Every induced subgraph G' of G is either an induced subgraph of $G - v$ or contains v. Hence, G' has a vertex of degree less than Δ contradicting the claim that G has degeneracy Δ. ■

Lemma 2.14 tells us that Theorem 2.7 is a statement about $(\Delta - 1)$-degenerate graphs.

We introduce some additional definitions. We let $G[S]$ denote the subgraph of a graph $G = (V, E)$ induced by some set $S \subseteq V$. It is well-known that G is p-degenerate for some integer p if and only if there exists a *degeneracy ordering* $v_1, v_2, \ldots, v_n$ of its vertices such that v_i has at most p neighbours v_j with $j < i$. One can compute such an ordering in $O(n^2)$ time (let v_n be a vertex of minimum degree in G and, for $i = n-1, \ldots, 1$, let v_i be a vertex of minimum degree in $G[V \setminus \{v_{i+1}, \ldots, v_n\}]$).

We need an algorithmic version of a result of Mihók [96], which was proven independently by Wood [118]. We present a slightly modified version of the proof of Wood which was implicitly algorithmic (it suffices to make a few additional algorithmic observations).

Lemma 2.15 ([96, 118])**.** *Let $r \geq 1$ and $k \geq r - 1$. Let $G = (V, E)$ be a k-degenerate graph on n vertices. Let $p_1, \ldots, p_r$ be non-negative integers so that $\sum_{t=1}^{r} p_t = k - r + 1$. Then it is possible to compute in $O(n^2)$ time a partition $\{V_1, \ldots, V_r\}$ of V such that $G[V_t]$ is p_t-degenerate for $t = 1, \ldots, r$.*

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Proof. We first compute a degeneracy ordering $v_1, \ldots, v_n$ of G in $O(n^2)$ time. For $i = 1, \ldots, n$, we define $X_i = \{v_1, \ldots, v_i\}$. Then, by definition, v_i has at most k neighbours in X_{i-1}. It suffices to show that, for $2 \leq i \leq n$, we can compute in $O(n)$ time a partition $\{Y_1, \ldots, Y_r\}$ of X_i, where $G[Y_s]$ is p_s-degenerate for $s = 1, \ldots, r$, if we have as input such a partition of X_{i-1}. We note first that finding a partition of X_1 is trivial. Let $i \geq 2$. Let $\{Z_1, \ldots, Z_r\}$ be a partition of X_{i-1} where $G[Z_s]$ is p_s-degenerate for $s = 1, \ldots, r$. If v_i has more than p_s neighbours in every $G[Z_s]$, then v_i has at least $\sum_{i=1}^{r}(p_i+1) = k+1$ neighbours in X_{i-1}, a contradiction. Hence, v_i has at most p_q neighbours in at least one set Z_q, which we can find in $O(n)$ time. We put v_i into Z_q to get the desired partition for X_i in $O(n)$ time. ■

Recall that Cereceda [25] proved that for any $k \geq 2d + 1$ the reconfiguration graph $R_k(G)$ of every d-degenerate graph G on n vertices has diameter $O(n^2)$. We adapt his proof to show the following lemma.

Lemma 2.16. *Let G be a graph on n vertices with maximum degree $\Delta \geq 1$ and degeneracy $\Delta - 1$. Let α be a $(\Delta + 1)$-colouring of G. It is possible to compute a Δ-colouring γ of G in time $O(n^2)$ such that $d_{\Delta+1}(\alpha, \gamma) \leq n^2$.*

Proof. We first compute a degeneracy ordering $v_1, \ldots, v_n$ of G in $O(n^2)$ time. Also in $O(n^2)$ time we record, for each vertex v, the neighbour of v that is latest in the ordering, and the set of colours that are not used on neighbours of v.

Let h be the lowest index such that $\alpha(v_h) = \Delta+1$. We will describe an algorithm that finds in time $O(n)$ a sequence of recolourings such that

- for $i < h$, v_i is not recoloured,
- for $i \geq h$, v_i is recoloured at most once, and
- v_h is recoloured with a colour other than $\Delta + 1$.

By repeatedly using such sequences, we can obtain a colouring γ in which colour $\Delta+1$ is not used. At most n such sequences are needed, so each vertex is recoloured at most n times and the lemma follows.

We must describe the algorithm. First we find a sequence S of pairs of vertices and colours (w_j, c_j) as follows:

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- the first vertex w_1 is v_h;
- for each vertex w_j, if there is a colour that is not used on it or any of its neighbours, then this is c_j, and (w_j, c_j) is the final pair in S;
- otherwise let w_{j+1} be the neighbour of w_j that is latest in the degeneracy ordering and let $c_j = \alpha(w_{j+1})$.

If all $\Delta+1$ colours appear on w_j and its neighbours, then w_j must have degree Δ, each neighbour of w_j must have a distinct colour, and, as at most $\Delta - 1$ neighbours can be earlier in the degeneracy ordering, at least one neighbour is later in the ordering. Thus each vertex in S is later in the degeneracy ordering than its predecessor and so the algorithm will terminate and S is finite. Moreover, this also implies that each vertex in $v_{h+1}, \dots, v_n$ is considered at most once during the computation of S and so, as the information required about each vertex was found during our preliminary computations, we can find S in $O(n)$ time.

Let s denote the number of pairs in S. We can recolour the vertices of S in time $O(n)$ by simply recolouring w_j with c_j, starting with w_s and working backwards through S. Each colouring obtained is proper since w_s has no neighbour coloured c_s and when a vertex v_j, $j < s$ is recoloured, its unique neighbour w_{j+1} coloured c_j has just been recoloured and it is not adjacent to any other vertex that has been recoloured since they are later in the degeneracy ordering than any of its neighbours. Finally note that $w_1 = v_h$ has been recoloured with a colour other than $\alpha(v_h) = \Delta + 1$, so the recolouring sequence achieves its aim: the index of the first vertex in the ordering coloured $\Delta + 1$ is now greater than h. This completes the proof. ■

Finally we need an algorithmic version of Lemma 2.13 for the special case of $(\Delta-1)$-degenerate graphs; to prove it we follow the line of the proof of Lemma 2.13, but need Lemma 2.15 instead of Lemma 2.11 and Lemma 2.16 instead of Lemma 2.12. The question whether there exists an algorithmic version for the remaining case of Δ-regular graphs is still open; note that one cannot replace Lemma 2.15 by Lemma 2.11 in the proof of Lemma 2.17, as that would require solving the NP-complete problem

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of finding a maximum independent set in graphs of bounded maximum degree in polynomial time.

Lemma 2.17. *Let $G = (V, E)$ be a $(\Delta - 1)$-degenerate graph on n vertices with maximum degree $\Delta \geq 1$. It is possible to find in $O(n^2)$ time a path between any two given Δ-colourings γ_1 and γ_2 in $R_{\Delta+1}(G)$.*

Proof. We use induction on Δ. If $\Delta \in \{1, 2\}$ the statement is trivially true. Let $\Delta \geq 3$ and assume that we have an $O(n^2)$-time algorithm for connected $(\Delta - 2)$-degenerate graphs on n vertices with maximum degree $\Delta - 1$.

Applying Lemma 2.15 with $p_1 = 0$ and $p_2 = \Delta - 2$ gives us in $O(n^2)$ time a partition $\{S_1, S_2\}$ of V such that S_1 is an independent set and S_2 induces a $(\Delta-2)$-degenerate graph that we denote by H. We modify the pair (S_1, S_2) in $O(n^2)$ time by moving vertices from S_2 to S_1 until S_1 is a maximal independent set. Let γ_1^H and γ_2^H be the colourings of H that are the restrictions of γ_1 and γ_2 to S_2. We note that

- γ_1^H and γ_2^H use only colours from $\{1, 2, \ldots, \Delta\}$;
- H has maximum degree at most $\Delta - 1$ (by the maximality of S_1);
- H is $(\Delta - 2)$-degenerate.

Thus we can apply Lemma 2.16 to recolour each of γ_1^H and γ_2^H to a $(\Delta-1)$-colouring in $O(n^2)$ time. We then apply the induction hypothesis to find in $O(n^2)$ time a path between these two $(\Delta - 1)$-colourings that includes only Δ-colourings. Hence the total running time is $O(n^2)$, as required. ■

We are now ready to prove Theorem 2.7, which we first restate.

Theorem 2.7. *Let G be a connected graph on n vertices with maximum degree $\Delta \geq 3$. Let $k = \Delta + 1$. If G is not regular, then it is possible to find in $O(n^2)$ time a path between any two given k-colourings α and β in $R_k(G)$.*

Proof. By Lemma 2.14 we find that G is $(\Delta - 1)$-degenerate. By Lemma 2.16 we can find in $O(n^2)$ time a path from α to some Δ-colouring γ_1 and a path from β to some Δ-colouring γ_2. Applying Lemma 2.17 completes the proof. ■

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2.5 Final Remarks

Let us mention an observation concerning the computational problem of deciding if a graph with maximum degree $\Delta \geq 3$ admits a frozen $(\Delta+1)$-colouring. We can formulate this problem in the language of graph homomorphism. Let G and H be graphs. A *homomorphism* from G to H is a function $f : V(G) \to V(H)$ that maps adjacent vertices of G to adjacent vertices of H; that is $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. In addition f is said to be *surjective* if for each $x \in V(H)$ there exists at least one vertex $v \in V(G)$ with $f(v) = x$. Further, f is said to be *locally surjective* or *strongly locally surjective* if, for every vertex v of G, f becomes surjective when restricted to respectively the open neighbourhood $N(v)$ or closed neighbourhood $N(v) \cup \{v\}$ of v. The following proposition follows easy from the definitions.

Proposition 2.18. *Let G be a graph with maximum degree $\Delta \geq 3$. Then G has a frozen $(\Delta+1)$-colouring if and only if G has a strongly locally surjective homomorphism to the complete graph $K_{\Delta+1}$.*

To the best of our knowledge, the complexity of deciding if a graph with maximum degree $\Delta \geq 3$ admits a strongly locally surjective homomorphism to $K_{\Delta+1}$ is not known. We conjecture that this problem is NP-complete since, for example, the problem of deciding if a graph has a locally surjective homomorphism to the complete graph on at least 3 vertices is NP-complete [56]. (Notice that in the latter result the input graph does not necessarily have bounded maximum degree).

We end this section with a final remark on Conjecture 2.5. It turns out that Conjecture 2.5 is equivalent to the following conjecture that is easier in appearance.

Conjecture 2.19. *Let G be a k-degenerate graph, and let α be a $(k+2)$-colouring of G. Then there exists a $(k+1)$-colouring γ of G such that $d_{k+2}(\alpha, \gamma) \leq O(n^2)$.*

Proposition 2.20. *Conjectures 2.5 and 2.19 are equivalent.*

Proof. As any k-degenerate graph admits a $(k+1)$-colouring, Conjecture 2.5 immediately implies Conjecture 2.19. For the other direction we use induction on the degeneracy k of G.

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If $k = 1$ Conjecture 2.5 is known to be true [25]. Assume now that the result holds for all graphs of degeneracy less than k. Let α and β be two $(k+2)$-colourings of G. Applying Conjecture 2.19 to α and β gives us $(k+1)$-colourings α' and β' respectively such that $d_{\Delta+2}(\alpha, \alpha') \leq O(n^2)$ and $d_{\Delta+2}(\beta, \beta') \leq O(n^2)$. Applying Lemma 2.15 with $p_1 = 0$ and $p_2 = k - 1$ we find a partition $\{S_1, S_2\}$ of V such that S_1 is an independent set and S_2 induces a $(k-1)$-degenerate graph that we denote H. From α' and β' recolour the vertices of S_1 with colour $k + 2$ (the colour that is not used in either α' or β'). This can be done by at most $2n$ recolourings. Denote by α'_H and β'_H the restrictions of α' and β' to H. By the induction hypothesis, there is a sequence of $O(n^2)$ recolourings from α'_H to β'_H without using colour $k+2$. This implies the proposition. ■

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Chapter 3

On a Conjecture of Mohar Concerning Kempe Equivalence of Regular Graphs

3.1 Introduction

We start by recalling some of the notation and terminology from Chapter 1. For a colouring α and colours a and b, $G_\alpha(a,b)$ is the subgraph of G induced by vertices with colour a or b under α. A connected component of $G_\alpha(a,b)$ is known as an (a,b)-*component under* α of G (we will omit the reference to α if the dependency on α is clear from the context). These components are also referred to as *Kempe chains*. If a colouring β is obtained from a colouring α by exchanging the colours a and b on the vertices of an (a,b)-component of G and α, then β is said to have been obtained from α by a *Kempe change*. Let $C_k(G)$ or C_G^k denote the set of all k-colourings of G. Two colourings $\alpha, \beta \in C_k(G)$ are *Kempe equivalent*, denoted by $\alpha \sim_k \beta$, if each can be obtained from the other by a series of Kempe changes. The equivalence classes $C_k(G)/\sim_k$ are called *Kempe classes*.

In this chapter we are concerned with a conjecture of Mohar which asserts that for all connected k-regular graphs that are not complete, the set of k-colourings form a Kempe class [97].

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Conjecture 3.1 ([97]). *Let $k \geq 3$. If G is a connected k-regular graph that is not K_{k+1} then $C_k(G)$ is a Kempe class.*

We prove that the conjecture holds with the only exception being the triangular prism depicted in Figure 3.1.

Theorem (Theorems 3.2 and 3.14 combined). *Let $k \geq 3$, and let G be a connected k-regular graph. If G is not $K_{\Delta+1}$, then $C_\Delta(G)$ is a Kempe class unless G is the triangular prism.*

Notice that we need not have included the condition that G is not complete since one can say that if a graph has no k-colourings, then this set of colourings (the empty set) is a Kempe class, but it is neater to exclude this case.

Let us describe another way to think of Conjecture 3.1. Let $\mathcal{K}_k(G)$ be the graph that has vertex set $C_k(G)$ and an edge between two vertices α and β whenever the colouring β can be obtained from α by a single Kempe change. Conjecture 3.1 states that, for $k \geq 4$, for any connected non-complete k-regular graph G, $\mathcal{K}_k(G)$ is connected.

We might call $\mathcal{K}_k(G)$ a *solution graph*; it represents all possible solutions to the problem of finding a k-colouring of G. Or we can call it the *reconfiguration graph* of k-colourings of G and refer to Kempe changes as reconfiguration steps. As mentioned in the preceding chapter, reconfiguration graphs of k-colourings have been much studied when the edge relation is defined by the alternative reconfiguration step of *trivial* Kempe changes where a Kempe chain is trivial if it contains a single vertex v and the corresponding Kempe change alters the colour of only v, and so pairs of colourings are connected in these reconfiguration graphs if they disagree on only one vertex.

Reconfiguration graphs defined by Kempe changes have received less attention. Kempe changes were introduced in 1879 by Kempe in his proof of the Four Colour Theorem [84]. Though this was fallacious, the Kempe change technique has proved useful in, for example, the proof of the Five Colour Theorem and a short proof of Brooks' theorem [92].

We review the purely graph theoretical studies of Kempe equivalence. Fisk [57] showed in 1977 that the 4-colourings of a Eulerian triangulation of the plane are

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Kempe equivalent. Afterwards Meyniel [94] showed that the 5-colourings of a planar graph are Kempe equivalent. The proof can be summarised as follows. Let v be a vertex of a planar graph G with degree at most 5 (such a vertex exists since planar graphs are 5-degenerate). Let α and β be 5-colourings of G and let $\{u, w\}$ and $\{x, z\}$ be pairs of neighbours of v that are coloured alike under α and β, respectively. Since the graphs G_1 and G_2 obtained from the graph $G - v$ by identifying the pairs $\{u, w\}$ and $\{x, z\}$ respectively are planar we can apply the induction hypothesis to the graph G_i to find that the set of 5-colourings of the graph G_i is a Kempe class for $i = 1, 2$. The proof ends by considering the cases $\{u, w\} = \{x, z\}$, $\{u, w\} \cap \{x, z\} = \emptyset$ and $|\{u, w\} \cap \{x, z\}| = 1$ independently. In the final section, we give a new proof of this result.

Meyniel's result was later extended by Mohar [97] who proved that the set of 4-colourings of a 3-colourable planar graph is a Kempe class. Mohar essentially shows that for a planar graph G and a 4-colouring α of G there exists a 4-colouring α' of G that is Kempe equivalent to α, a supergraph G' of G such that G' is a triangulation of the plane, and that α' is extendible to a k-colouring of G'. Since the k-colourings of G' are known to be Kempe equivalent by the aforementioned result of Fisk [57], it follows with the aid of an easy lemma that the k-colourings of G are Kempe equivalent.

Las Vergnas and Meyniel [114] showed that the set of 5-colourings of a K_5-minor free graph is a Kempe class. Their result is an extension (rather than a generalisation) of Meyniel's result since the proof relies on Wagner's well-known characterization of K_5-minor free graphs. The paper uses the powerful methods developed in [94] such as vertex identification and reverse induction on the number of edges. It also contains a number of interesting observations and conjectures related to Hadwiger's conjecture.

Bertschi [5] proved that the set of k-colourings of a perfectly contractile graph is a Kempe class (a perfectly contractile graph is defined recursively as either the complete graph or the graph having a pair of vertices x, y such that every induced path from x to y has even length and the graph obtained by identifying x, y into a new vertex is also perfectly contractile).The proof is straightforward and only

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constitutes a fraction of the paper, which chiefly aims at showing that perfectly contractile graphs are perfect. Meyniel [95] conjectured that the k-colourings of a Meyniel graph are Kempe equivalent. Since Meyniel graphs are perfectly contractile, Bertschi's result settled this conjecture in the affirmative. The Kempe equivalence of edge-colourings has also been investigated [3, 91, 97].

From a practical viewpoint, the Kempe change method has proved to be a powerful tool with applications to several areas such as timetables [104], theoretical physics [116, 117], and Markov chains [115]. The reader is referred to [97, 108] for further details.

3.2 Cubic Graphs

In this section, we address Conjecture 3.1 for the case $k = 3$. For this case the conjecture is known to be false. A counter-example is the 3-prism displayed in Figure 3.1. The fact that some 3-colourings of the 3-prism are not Kempe equivalent was already observed by van den Heuvel [113]. Our contribution is that the 3-prism is the *only* counter-example for the case $k = 3$, that is, we completely settle the case $k = 3$ by proving the following result for 3-regular graphs also known as *cubic* graphs.

Theorem 3.2. *If G is a cubic graph that is neither K_4 nor the 3-prism, then $C_3(G)$ is a Kempe class.*

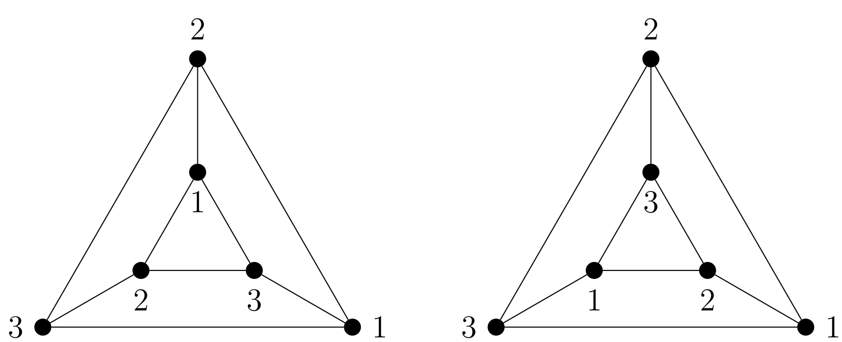

Figure 3.1: The triangular prism with two non-Kempe-equivalent 3-colourings.

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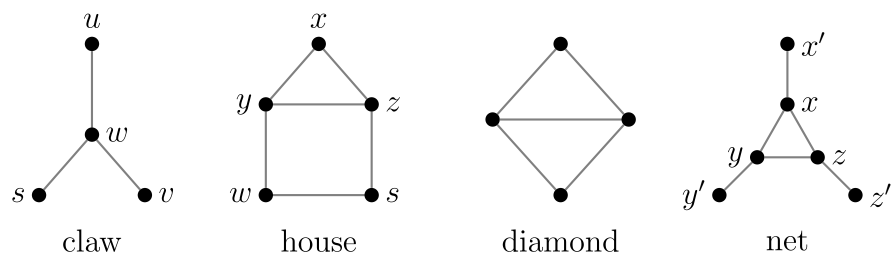

Figure 3.2: A number of special graphs used in Section 3.2.

3.2.1 The Proof of Theorem 3.2

We first give some further definitions and terminology.A *separator* of G is a set $S \subset V$ such that $G - S$ has more components than G. We say that G is *p-connected* for some integer p if $|V| \geq p + 1$ and every separator of G has size at least p. Some small graphs that we will refer to are defined by their illustrations in Figure 3.2.

Besides three new lemmas, we will need the aforementioned result of van den Heuvel, which follows from the fact that for the 3-prism T, the subgraphs $T(1, 2)$, $T(2, 3)$ and $T(1, 3)$ are connected so that the number of Kempe classes is equal to the number of different 3-colourings of T up to colour permutation, which is two.

Lemma 3.3 ([113])**.** *If G is the 3-prism, then $C_3(G)$ consists of two Kempe classes.*

Lemma 3.4. *If G is a cubic graph that is connected but not 3-connected, then $C_3(G)$ is a Kempe class.*

Lemma 3.5. *If G is a 3-connected cubic graph that is claw-free but that is neither K_4 nor the 3-prism, then $C_3(G)$ is a Kempe class.*

Lemma 3.6. *If G is a 3-connected cubic graph that is not claw-free, then $C_3(G)$ is a Kempe class.*

Observe that Theorem 3.2 follows from the above lemmas, which form a case distinction. Hence it suffices to prove Lemmas 3.4–3.6. These proofs form the remainder of this section.

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Proof of Lemma 3.4

In order to prove Lemma 3.4 we need three auxiliary results and one more definition: recall that a graph G is *d-degenerate* if every induced subgraph of G has a vertex with degree at most d.

Lemma 3.7 ([114, 97])**.** *Let d and k be any two integers with $d \geq 0$ and $k \geq d+1$. If G is a d-degenerate graph, then $C_k(G)$ is a Kempe class.*

Lemma 3.8 ([114])**.** *Let $k \geq 1$ be an integer. Let G_1, G_2 be two graphs such that $G_1 \cap G_2$ is complete. If both $C_k(G_1)$ and $C_k(G_2)$ are Kempe classes, then $C_k(G_1 \cup G_2)$ is a Kempe class.*

Lemma 3.9 ([97])**.** *Let $k \geq 1$ be an integer and let G be a subgraph of a graph G'. Let c_1 and c_2 be the restrictions, to G, of two k-colourings c_1' and c_2' of G'. If c_1' and c_2' are Kempe equivalent, then c_1 and c_2 are Kempe equivalent.*

For convenience we restate Lemma 3.4 before we present its proof.

Lemma 3.4 (restated). *If G is a cubic graph that is connected but not 3-connected, then $C_3(G)$ is a Kempe class.*

Proof. As disconnected graphs can be considered component-wise, we assume that G is connected. As G is cubic, G has at least four vertices. Because G is not 3-connected, G has a separator S of size at most 2. Let S be a minimum separator of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = S$. As every vertex in S has degree at most 2 in each G_i and G is cubic, G_1 and G_2 are 2-degenerate. Hence, by Lemma 3.7, $C_3(G_1)$ and $C_3(G_2)$ are Kempe classes. If S is a clique, we apply Lemma 3.8. Thus we assume that S, and any other minimum separator of G, is not a clique. Then $S = \{x, y\}$ for two distinct vertices x and y with $xy \notin E(G)$.

Because S is a minimum separator, x and y are non-adjacent and G is cubic, x has either one neighbour in G_1 and two in G_2, or the other way around; the same holds for vertex y. For $i = 1, 2$, let $N_i(x)$ and $N_i(y)$ be the set of neighbours of x and y, respectively, in G_i. Then we have that either $|N_1(x)| = 1$ and $|N_2(x)| = 2$, or $|N_1(x)| = 2$ and $|N_2(x)| = 1$, and similarly, that either $|N_1(y)| = 1$ and $|N_2(y)| = 2$, or $|N_1(y)| = 2$ and $|N_2(y)| = 1$. Let $x_1 \in N_1(x)$ for some $x_1 \in V(G_1)$.

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We may assume that $|N_1(x)| \neq |N_1(y)|$; if not we can do as follows. Assume without loss of generality that $N_1(x) = \{x_1\}$ and that $|N_1(y)| = 1$. Then $\{x_1, y\}$ is a separator. By our assumption that G has no minimum separator that is a clique, we find that $\{x_1, y\}$ is a minimum separator with $x_1y \notin E(G)$. As G is cubic, x_1 has two neighbours in $V(G_1) \setminus \{x, x_1\}$. As $|N_1(y)| = 1$ and x_1 and y are not adjacent, y has exactly one neighbour in $V(G_1) \setminus \{x, x_1\}$. Hence we could take $\{x_1, y\}$ as our minimum separator instead of S in order to get the desired property. We may thus assume that $|N_1(x)| \neq |N_1(y)|$. As this means that $|N_2(x)| \neq |N_2(y)|$, we can let $N_1(x) = \{x_1\}$ and $N_2(y) = \{y_1\}$ for some $y_1 \in V(G_2)$.

It now suffices to prove the following two claims.

Claim 1. All colourings α such that $\alpha(x) \neq \alpha(y)$ are Kempe equivalent in $C_3(G)$.

We prove Claim 1 as follows. We add an edge e between x and y. This results in graphs $G_1 + e$, $G_2 + e$ and $G + e$. We first prove that $C_3(G + e)$ is a Kempe class. Because x and y have degree 1 in G_1 and G_2, respectively, and G is cubic, we find that the graphs $G_1 + e$ and $G_2 + e$ are 2-degenerate. Hence, by Lemma 3.7, $C_3(G_1+e)$ and $C_3(G_2+e)$ are Kempe classes. By Lemma 3.8, it holds that $C_3(G+e)$ is a Kempe class. Applying Lemma 3.9 completes the proof of Claim 1.

Claim 2. For every colouring α such that $\alpha(x) = \alpha(y)$, there exists a colouring β with $\beta(x) \neq \beta(y)$ such that α and β are Kempe equivalent in $C_3(G)$.

We assume without loss of generality that $\alpha(x) = \alpha(y) = 1$ and $\alpha(y_1) = 2$. If $\alpha(x_1) = 2$, then we apply a Kempe change on the $(1, 3)$-component of G that contains x. Note that y does not belong to this component. Hence afterwards we obtain the desired colouring γ. If $\alpha(x_1) = 3$, then we first apply a Kempe change on the $(2, 3)$-component of G that contains x_1. Note that this does not affect the colours of x, y and y_1 as they do not belong to this component. Afterwards we proceed as before. This completes the proof of Claim 2 (and the lemma). ■

Proof of Lemma 3.5

We require some further terminology and three lemmas. Two colourings α and β of a graph G *match* if there exists two vertices x, y with a common neighbour in G

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such that $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$.

Lemma 3.10. *Let $k \geq 1$ and G' be the graph obtained from a graph G by identifying two non-adjacent vertices x and y. If $C_k(G')$ is a Kempe class, then all k-colourings c of G with $c(x) = c(y)$ are Kempe equivalent.*

Proof. Let α and β be two k-colourings of G with $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$. Let z be the vertex of G' that is obtained after identifying x and y. Let α' and β' be the k-colourings of G' that agree with α and β, respectively, on $V(G) \setminus \{x, y\}$ and for which $\alpha'(z) = \alpha(x)(= \alpha(y))$ and $\beta'(z) = \beta(x)(= \beta(y))$. By our assumption, there exists a Kempe chain from α' to β' in G'. We mimic this Kempe chain in G. Note that any (a, b)-component in G' that contains z corresponds to at most two (a, b)-components in G, as x and y may get separated. Hence, every Kempe change on an (a, b)-component corresponds to either one or two Kempe changes in G (if x and y are in different (a, b)-components, then we apply the corresponding Kempe change in G' on each of these two components). In this way we obtain a Kempe chain from α to β as required. ■

Lemma 3.11. *Let $k \geq 3$. If α and β are matching k-colourings of a 3-connected graph G of maximum degree k, then $\alpha \sim_k \beta$.*

Proof. If G is $(k-1)$-degenerate, then $\alpha \sim_k \beta$ by Lemma 3.7. Assume that G is not $(k-1)$-degenerate. Then G is k-regular. Since α and β match, there exist two vertices u and v of G that have a common neighbour w such that $\alpha(u) = \alpha(v)$ and $\beta(u) = \beta(v)$. Let x denote the vertex of G' obtained by identifying u and v.

Let S be a separator of G'. If S does not contain x, then S is a separator of G. Then $|S| \geq 3$ as G is 3-connected. If S contains x, then S must contain another vertex as well; otherwise $\{u, v\}$ is a separator of size 2 of G, which is not possible. Hence, $|S| \geq 2$ in this case. We conclude that G' is 2-connected.

We now prove that G' is $(k-1)$-degenerate. Note that, in G', w has degree $k-1$, x has degree at least k and all other vertices have degree k. Let $u_1, \ldots, u_r$ for some $r \geq k-1$ be the neighbours of x not equal to w. Since G' is 2-connected, the graph $G'' = G' \backslash x$ is connected. This means that every u_i is connected to w via a path in G'', which corresponds to a path in G' that does not contain x. Since w has degree

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$k-1$ and every vertex not equal to x has degree k, we successively delete vertices of these paths starting from w towards u_i so that each time we delete a vertex of degree at most $k-1$. Afterwards we can delete x as x has degree 0. The remaining vertices form an induced subgraph of G' whose components each have maximum degree at least k and at least one vertex of degree at most $k-1$. Hence, we can continue deleting vertices of degree at most $k-1$ and thus find that G' is $(k-1)$-degenerate. Then, by Lemma 3.7, $C_k(G')$ is a Kempe class. Hence, by Lemma 3.10, we find that $\alpha \sim_k \beta$ as required. This completes the proof. ■

Lemma 3.12. *Every 3-connected cubic claw-free graph G that is neither K_4 nor the 3-prism is house-free, diamond-free and contains an induced net (see also Figure 3.2).*

Proof. First suppose that G contains an induced diamond D. Then, since G is cubic, the two non-adjacent vertices in D form a separator and G is not 3-connected, a contradiction. Consequently, G is diamond-free.

Now suppose that G contains an induced house H. We use the vertex labels of Figure 3.2. So, s, w, x are the vertices that have degree 2 in H, and s and w are adjacent. As G is cubic, w has a neighbour $t \in V(G) \setminus V(H)$. Since G is cubic and claw-free, t must be adjacent to s. If $tx \in E$, then G is the 3-prism. If $tx \notin E$, then t and x form a separator of size 2. In either case we have a contradiction. Consequently, G is house-free.

We now prove that G has an induced net. As G is cubic and claw-free, it has a triangle and each vertex of the triangle has one neighbour in G outside the triangle. Because G is not K_4 and diamond-free, these neighbours are distinct. Then, because G is house-free, no two of them are adjacent. Hence, together with the vertices of the triangle, they induce a net. ■

We restate Lemma 3.5 before we present its proof.

Lemma 3.5 (restated). *If G is a 3-connected cubic graph that is claw-free but that is neither K_4 nor the 3-prism, then $C_3(G)$ is a Kempe class.*

Proof. By Lemma 3.12, G contains an induced net N. For the vertices of N we use the labels of Figure 3.2. In particular, we refer to x, y and z as the *t-vertices*

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of N, and x', y' and z' as the *p-vertices*. Let α and β be two 3-colourings of G. In order to show that $\alpha \sim_3 \beta$ we distinguish two cases.

Case 1. There are two p-vertices with identical colours under α or β.
Assume that $\alpha(x') = \alpha(y') = 1$. Then $\alpha(z) = 1$ as the t-vertices form a triangle, so colour 1 must be used on one of them. Assume without loss of generality that $\alpha(z') = \alpha(x) = 2$ and so $\alpha(y) = 3$. If $\beta(z') = \beta(x)$, then α and β match (as x and z' have z as a common neighbour). Then, by Lemma 3.11, we find that $\alpha \sim_3 \beta$. Otherwise $\beta(z') = \beta(y)$, since the colour of z' must appear on one of x and y. Note that the $(2, 3)$-component containing x under α consists only of x and y. Then a Kempe exchange applied to this component yields a colouring α' such that $\alpha'(z') = \alpha'(y)$. As y and z' have z as a common neighbour as well, this means that α' and β match. Hence, it holds that $\alpha \sim_3 \alpha' \sim_3 \beta$, where the second equivalence follows from Lemma 3.11.

Case 2. All three p-vertices have distinct colours under both α and β.
Assume without loss of generality that $\alpha(x) = \alpha(z') = 1$, $\alpha(y) = \alpha(x') = 2$, and $\alpha(z) = \alpha(y') = 3$. Note that Kempe chains of G are paths or cycles, as no vertex in a chain can have degree 3 since all its neighbours in a chain are coloured alike and G is claw-free. So, we will refer to (a, b)-*paths* rather than (a, b)-components.

We will now prove that there exists a colouring α' with $\alpha \sim_3 \alpha'$ that assigns the same colour to two p-vertices of N. This suffices to complete the proof of the lemma, as afterwards we can apply Case 1.

Consider the $(1, 2)$-path P that contains x'. If P does not contain z', then a Kempe exchange on P gives us a desired colouring α' (with x' and z' coloured alike). So we can assume that x' and z' are joined by a $(1, 2)$-path P_{12}, and, similarly, x' and y' by a $(2, 3)$-path P_{23}, and y' and z' by a $(1, 3)$-path P_{13}.

Let G' be the subgraph of G induced by the three paths. Note that P_{12} has end-vertices y and z', P_{23} has end-vertices z and x' and P_{13} has end-vertices x and y'. Hence, G' contains the vertices of N and every vertex in $G' - N$ is an internal vertex of one of the three paths. As G is cubic, this means that each vertex in $G' - N$ belongs to exactly one path. Moreover, as G is claw-free and cubic, two vertices in $G' - N$ that are on different paths are adjacent if and only if they have a p-vertex

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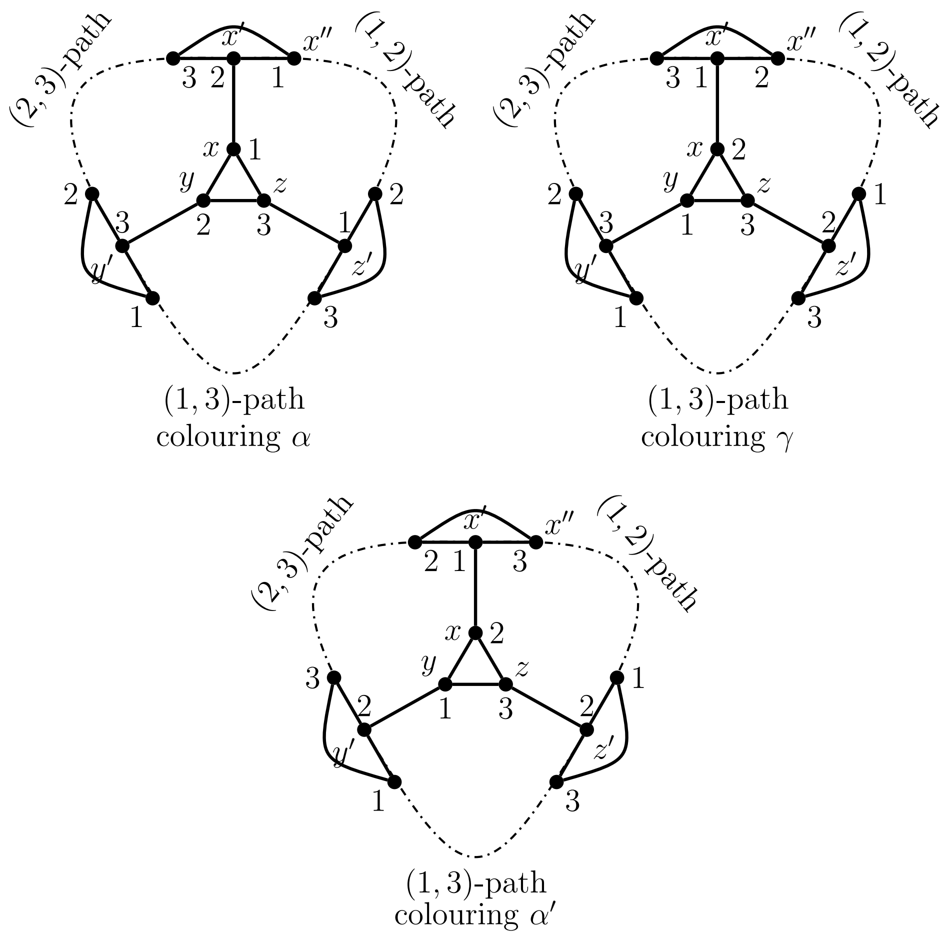

Figure 3.3: Colourings of G' in the proof of Lemma 3.5. The dotted lines indicate paths of arbitrary length.

as a common neighbour.

In Figure 3.3 are illustrations of G' and the colourings of this proof that we are about to discuss. Let $x'' \neq x$ be the vertex in P_{12} adjacent to x'. From the above it follows that x'' is adjacent to the neighbour of x' on P_{23} and that no other vertex of P_{12} (apart from x') is adjacent to a vertex of P_{23}. As G is cubic, this also means that x'' has no neighbour outside G'. Apply a Kempe exchange on P_{12} and call the resulting colouring γ. By the arguments above, the new $(2,3)$-path Q_{23} (under γ) that contains y' has vertex set $(V(P_{23}) \cup \{x''\}) \setminus \{x', y, z\}$. Apply a Kempe exchange on Q_{23}. This results in a colouring α' with $\alpha'(y') = \alpha'(z') = 2$, hence α' is a desired colouring. This completes the proof of Case 2 and thus of the lemma. ■

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Proof of Lemma 3.6

We first need another lemma.

Lemma 3.13. *Let W be a set of three vertices with a common neighbour in a 3-connected cubic graph G. Suppose that every 3-colouring γ of G that colours alike exactly one pair of W is Kempe equivalent to a 3-colouring γ' such that γ' colours alike a different pair of W. Then $C_3(G)$ is a Kempe class.*

Proof. Let α and β be two 3-colourings of G. To prove the lemma we show that $\alpha \sim_3 \beta$. By Lemma 3.11, it is sufficient to find a matching pair of colourings that are Kempe equivalent to α and β respectively (this lemma will be applied repeatedly).

As the three vertices of W have a common neighbour, in any 3-colouring at least two of them are coloured alike. Let $W = \{x, y, z\}$. We can assume that $\alpha(x) = \alpha(y)$. If $\beta(x) = \beta(y)$, then α and β match and we are done. So we can instead assume that $\beta(x) \neq \beta(y)$ and thus $\beta(y) = \beta(z)$. If $\alpha(y) = \alpha(z)$, then, again, α and β match. Otherwise α colours alike exactly one pair of W and, by the premise of the lemma, we can find a 3-colouring α' that is Kempe equivalent to α and colours alike a different pair of W. If $\alpha'(y) = \alpha'(z)$, then α' and β match. Otherwise we must have that $\alpha'(x) = \alpha'(z)$. As $\beta(x) \neq \beta(y)$ and $\beta(y) = \beta(z)$, there exists a 3-colouring β' that is Kempe equivalent to β and that colours alike a different pair of W than β. So $\beta'(x) \in \{\beta'(y), \beta'(z)\}$ and β' matches either α or α'. In both cases we are done. ∎

We restate Lemma 3.6 before we present its proof.

Lemma 3.6 (restated). *If G is a 3-connected cubic graph that is not claw-free, then $C_3(G)$ is a Kempe class.*

Proof. Note that if a vertex has three neighbours coloured alike it is a single-vertex Kempe chain. We will write that such a vertex can be recoloured to refer to the exchange of such a chain.

We make repeated use of Lemma 3.11: two colourings are Kempe equivalent if they match.

Let C be a claw in G with vertex labels as in Figure 3.2. Note that in every 3-colouring of G, two of s, u and v are coloured alike since they have a common

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neighbour. If some fixed pair of u, v and s is coloured alike by every 3-colouring of G, then every pair of colourings matches and we are done. So let α be a 3-colouring of G and assume that $\alpha(u) = \alpha(v) = 1$ and that there are colourings for which u and v have distinct colourings, or, equivalently, colourings for which s has the same colour as either u or v. By Lemma 3.13, it is sufficient to find such a 3-colouring that is Kempe equivalent to α. Our approach is to divide the proof into a number of cases, and, in each case, start from α and make a number of Kempe changes until a colouring in which s agrees with either u or v is obtained. We will denote such a colouring ω to indicate a case is complete.

First some simple observations. If $\alpha(s) = 1$, then let $\omega = \alpha$ and we are done. So we can assume instead that $\alpha(s) = 2$ (and so, of course, $\alpha(w) = 3$). If it is possible to recolour one of u, v or s, then we can let ω be the colouring obtained. Thus we can assume now that each vertex of u, v and s has two neighbours that are not coloured alike.

For a colouring c, vertex x, and colours a and b let $F^{ab}_{c,x}$ denote the (a, b)-component at s under c. We can assume that $F^{12}_{\alpha,s}$ contains both u and v as otherwise exchanging $F^{12}_{\alpha,s}$ results in a colouring in which s agrees with either u or v.

Let $N(u) = \{w, u_1, u_2\}$, $N(v) = \{w, v_1, v_2\}$, and $N(s) = \{w, s_1, s_2\}$. Note that the vertices $u_1, u_2, v_1, v_2, s_1, s_2$ are not necessarily distinct.

Case 1. $\alpha(u_1) \neq \alpha(u_2)$, $\alpha(v_1) \neq \alpha(v_2)$ and $\alpha(s_1) \neq \alpha(s_2)$.
So each of u, v and s has degree 1 in $F^{12}_{\alpha,s}$ and therefore $F^{12}_{\alpha,s}$ has at least one vertex of degree 3. Let x be the vertex of degree 3 in $F^{12}_{\alpha,s}$ that is closest to u and let α' be the colouring obtaining by recolouring x. Then u is not in $F^{12}_{\alpha',s}$ which can be exchanged to obtain ω.

Case 2. $\alpha(s_1) = \alpha(s_2)$.
Then $\alpha(s_1) = \alpha(s_2) = 1$ else ω can be obtained by recolouring s.

Subcase 2.1: $\alpha(u_1) = \alpha(u_2)$ or $\alpha(v_1) = \alpha(v_2)$.
The two cases are equivalent so we consider only the first. We have $\alpha(u_1) = \alpha(u_2) = 2$ else u is not in $F^{12}_{\alpha,s}$. Note that $F^{23}_{\alpha,s}$ consists only of s and w. If $F^{23}_{\alpha,s}$ is exchanged, u has three neighbours coloured 2, and can be recoloured to obtain ω (as u and s are both now coloured 3).

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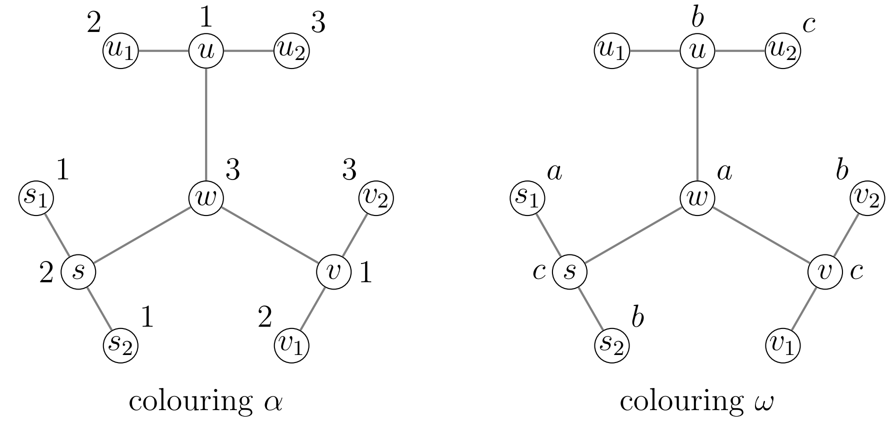

Figure 3.4: The colourings of Subcase 2.2 of Lemma 3.6.

Subcase 2.2: $\alpha(u_1) \neq \alpha(u_2)$ and $\alpha(v_1) \neq \alpha(v_2)$.
We can assume that $\alpha(u_1) = \alpha(v_1) = 2$, and $\alpha(u_2) = \alpha(v_2) = 3$.

In this case, we take a slightly different approach. Let ω now be some fixed 3-colouring with $\omega(s) \in \{\omega(u), \omega(v)\}$.We show that $\alpha \sim_3 \omega$ by making Kempe changes from α until a colouring that matches ω (or a colouring obtained from ω by a Kempe change) is reached.

Let $\{a, b, c\} = \{1, 2, 3\}$. If $\omega(s_1) = \omega(s_2)$, then ω matches α (recall $\alpha(s_1) = \alpha(s_2)$ in this case). So assume that $\omega(s_1) = a$ and $\omega(s_2) = b$. Then $\omega(s) = c$, and we can assume, without loss of generality, that $\omega(w) = a$. Note that we can assume that $\omega(u) \neq \omega(v)$ else α and ω match and we are done. So, as u and v are symmetric under α, we can assume that $\omega(u) = b$ and $\omega(v) = c$. If $\omega(u_2) = a$ or $\omega(v_2) = a$, then, again, α and ω match (recall that $\alpha(w) = \alpha(u_2) = \alpha(v_2)$) so we assume otherwise (noting that this implies $\omega(u_2) = c$ and $\omega(v_2) = b$) and consider two cases. For convenience, we first illustrate our current knowledge of α and ω in Figure 3.4. (Though it is not pertinent in this case, we again observe that the six vertices of degree 1 in the illustraton might not, in fact, be distinct.)

Subcase 2.2.1: $\omega(w) = a \in \{\omega(u_1), \omega(v_1)\}$.
Notice that $F^{23}_{\alpha,s}$ contains only s and w. If it is exchanged, then a colouring is obtained where w, u_1 and v_1 are coloured alike and this colouring matches ω.

Subcase 2.2.2: $\omega(w) = a \notin \{\omega(u_1), \omega(v_1)\}$.

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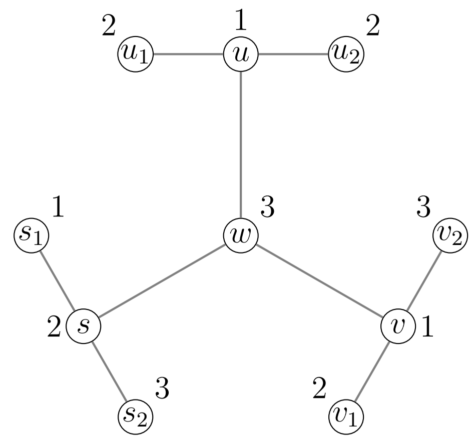

Figure 3.5: The colouring α of Case 3 of Lemma 3.6.

So $\omega(u_1) = c$ and $\omega(v_1) = b$. Thus $F^{ab}_{\omega,w}$ contains only u and w, and the colouring obtained by its exchange matches α as w and v_1 are both coloured b.

Case 3. $\alpha(u_1) = \alpha(u_2)$, $\alpha(v_1) \neq \alpha(v_2)$, and $\alpha(s_1) \neq \alpha(s_2)$.
If $\alpha(u_1) = \alpha(w)$, then the three neighbours of u are coloured alike and it can be recoloured to obtain ω. So suppose $\alpha(u_1) = \alpha(u_2) = 2$. We may assume that $\alpha(s_1) = 1$, $\alpha(s_2) = \alpha(v_2) = 3$, and $\alpha(v_1) = 2$; see the illustration of Figure 3.5.

We continue to assume that $F^{12}_{\alpha,s}$ contains u and v and note that s and v have degree 1 therein.

Subcase 3.1: $F^{12}_{\alpha,s}$ is not a path.
Let t be vertex of degree 3 closest to s in $F^{12}_{\alpha,s}$. Then t can be recoloured to obtain a colouring α' such that $F^{12}_{\alpha',s}$ does not contain v. Exchanging $F^{12}_{\alpha',s}$, we obtain ω.

Subcase 3.2: $F^{12}_{\alpha,s}$ is a path.
Note that $F^{12}_{\alpha,s}$ is a path from s to v through s_1 and u.

Subcase 3.2.1: F^{13}_{α,s_2} is a path from s_1 to s_2.
Note that $F^{13}_{\alpha,u} \neq F^{13}_{\alpha,s_2}$ since if $F^{13}_{\alpha,u}$ is a path, then u would be an endvertex coloured 1 implying $u = s_1$ contradicting that C is a claw. As G is cubic a vertex can belong to both $F^{12}_{\alpha,s}$ and F^{13}_{α,s_2} if it is an endvertex of one of them, and we note that s_1 is the only such vertex.

Let α' be the colouring obtained from α by the exchange of F^{13}_{α,s_2}. If $s \notin F^{12}_{\alpha',v}$, then let ω be the colouring obtained by the further exchange of $F^{12}_{\alpha',v}$.

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Otherwise, $F^{12}_{\alpha',v} = F^{12}_{\alpha',s}$, s and v each have degree 1 therein, and we can assume it is a path (else, as in Subcase 3.1, there is a vertex of degree 3 that can be recoloured to obtain α'' and $F^{12}_{\alpha'',s}$ does not contain v and can be exchanged to obtain ω). We can also assume that $F^{12}_{\alpha',s}$ contains $F^{12}_{\alpha,s} \setminus \{s_1\}$: if not, then $F^{13}_{\alpha,s_2} \setminus \{s_1, s_2\} \cap F^{12}_{\alpha,v} \neq \emptyset$ (recall that $F^{12}_{\alpha,s}$ is a path from s to v through s_1 and u), but their common vertices would have degree 4. Thus, in particular, $F^{12}_{\alpha',s}$ contains u and the vertex t at distance 2 from s in $F^{12}_{\alpha,s}$.

As t is not an endvertex in $F^{12}_{\alpha',s}$, s_1 is its only neighbour coloured 3 under α'. So $F^{23}_{\alpha',w}$ contains four vertices: w, s, s_1 and t. Let α'' be the colouring obtained from α' by the exchange of $F^{23}_{\alpha',w}$. If $t \notin \{u_1, u_2\}$, then u has three neighbours with colour 2 with α'' and so can be recoloured to obtain ω. Otherwise the conditions of Case 1 are now met.

Subcase 3.2.2: F^{13}_{α,s_2} is not a path from s_1 to s_2.

If $s_1 \notin F^{13}_{\alpha,s_2}$, then the exchange of F^{13}_{α,s_2} gives a colouring in which s_1 and s_2 are coloured alike (the colour of s is not affected by the exchange and either both or neither of u and v change colour). Thus Case 2 can now be used.

So we can assume that $s_1 \in F^{13}_{\alpha,s_2}$ has degree 1 in F^{13}_{α,s_2} (recall that s_1 has degree 2 in $F^{12}_{\alpha,s}$). If s_2 has degree 2 in F^{13}_{α,s_2}, then $F^{23}_{\alpha,s}$ contains only w, s and s_2. If it is exchanged, u has three neighbours with colour 2 and can be recoloured to ω.

Thus s_1 and s_2 both have degree 1 in F^{13}_{α,s_2}. Let x be the vertex of F^{13}_{α,s_2} closest to s_2. Then x can be recoloured to obtain a colouring α' such that F^{13}_{α',s_2} does not contain s_1. Exchanging F^{13}_{α',s_2} again takes us to Case 2. This completes Case 3.

By symmetry, we are left to consider the following case to complete the proof of the lemma.

Case 4. $\alpha(u_1) = \alpha(u_2)$, $\alpha(v_1) = \alpha(v_2)$, and $\alpha(s_1) \neq \alpha(s_2)$.

If $\alpha(v_1) = \alpha(v_2) = 3$, then v can be recoloured to obtain ω. So we can assume that $\alpha(v_1) = \alpha(v_2) = 2$, and, similarly, that $\alpha(u_1) = \alpha(u_2) = 2$. We can also assume that $F^{23}_{\alpha,s}$ is a path since otherwise the vertex of degree 3 closest to s can be recoloured. Define $S = \{u_1, u_2, v_1, v_2\}$. We distinguish two cases.

Subcase 4.1: $|S \cap F^{23}_{\alpha,s}| \geq 2$.

As $F^{23}_{\alpha,s}$ is a path and w is an endvertex, one vertex of S, say v_1, has degree 2 in

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$F^{23}_{\alpha,s}$. Consider $F^{13}_{\alpha,w}$: it consists only of vertices w, u, and v. After it is exchanged, v_1 has three neighbours with colour 3 and recolouring v_1 allows us to apply Case 3.

Subcase 4.2: $|S \cap F^{23}_{\alpha,s}| \leq 1$.

It follows, without loss of generality, that $\{u_1, u_2\} \cap F^{23}_{\alpha,s} = \emptyset$. Exchange F^{23}_{α,u_1} and F^{23}_{α,u_2} (which might be two distinct components or just one) to obtain a colouring α'. As $w \in F^{23}_{\alpha',s}$ (and hence $w \notin F^{23}_{\alpha,u_1} \cup F^{23}_{\alpha,u_2}$), every neighbour of u is coloured 3 and it can be recoloured to obtain ω. This completes Case 4 and the proof of Lemma 3.6. ■

3.3 Regular Graphs with Degree at least 4

In this section we affirm the conjecture for larger k.

Theorem 3.14. *Let $k \geq 4$ be a positive integer. If G is a connected non-complete k-regular graph, then the set of k-colourings of G is a Kempe class.*

Let us note an immediate corollary of our result.

Corollary 3.15. *Let G be a connected graph with maximum degree at most $k \geq 3$. Then $C_k(G)$ is a Kempe class unless G is the complete graph K_{k+1} or the 3-prism.*

Proof. A connected graph with maximum degree $k \geq 3$ is either k-regular or $(k-1)$-degenerate. The corollary follows from Theorems 3.2 and 3.14 and Lemma 3.7. ■

Our result implies that the Wang-Swendsen-Kotecký (WSK) algorithm for the zero-temperature q-state Potts antiferromagnet is ergodic on the triangular lattice whenever $q = 6$ and on the kagomé lattice whenever $q = 4$, thus answering some of the questions raised in [98, 99]. We discuss this in Section 3.3.1.

In Section 3.3.2 we introduce some useful lemmas. In the final section, Section 3.3.3, we prove Theorem 3.14. We conclude this section with some final comments on our investigations towards proving Theorem 3.14. By Lemma 3.3, we now know that, for $k \geq 3$, the only non-complete connected k-regular graph whose k-colourings are not a Kempe class is the triangular prism. So one might have hoped to find a counterexample to Theorem 3.14 by finding, for some $k \geq 4$, a connected

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non-complete k-regular graph with a k-colouring such that all Kempe changes maintain the colour partition. However, it is not hard to convince oneself that such a graph does not exist. Indeed, let us consider such a graph G and k-colouring α and obtain a contradiction. As G has more than k vertices some colour a appears on more than one vertex. If a colour b does not appear on any vertex, then changing the colour of one vertex from a to b gives a colouring with a different partition. And if b appears on only one vertex u, then changing the colour of a vertex not adjacent to u to b again changes the partition. So every colour appears on at least two vertices. If, for any vertex u, there is a colour other than $\alpha(u)$ that does not appear in its neighbourhood, then another trivial Kempe change gives a colouring with a different partition; so on the k vertices in the neighbourhood of u, one colour appears twice and every other colour but $\alpha(u)$ appears once. For every pair of colours a and b, $G_\alpha(a, b)$ is connected (else a Kempe change of one component gives a different partition). As every vertex in $G_\alpha(a, b)$ has degree 1 or 2, it is either a path or a cycle. As there are at least two vertices coloured a, there is a vertex u coloured c that has degree 2 in $G(a, c)$. Similarly there is a vertex v coloured c that has degree 2 in $G_\alpha(b, c)$; clearly $u \neq v$. Notice that u and v must both have degree 1 in $G_\alpha(c, d)$; that is $G_\alpha(c, d)$ is a path whose endvertices are both coloured c. But, by the same argument, $G(c, d)$ is a path whose endvertices are both coloured d. This contradiction proves that such a G and α cannot exist.

3.3.1 Ergodicity of the WSK algorithm

The q-state Potts model [105, 119, 120] is one of the simplest and most studied models in statistical mechanics with various applications from the theory of critical phenomena to condensed-matter systems. The model uses a finite graph $G = (V, E)$ where each vertex $v \in V$ is assigned a *spin* $\sigma(v) \in \{1, \dots, q\}$ (that is, the spins provide a not necessarily proper vertex-colouring of G). Furthermore, the q-state Potts models are dynamic models where the spin state of a vertex can be modified over time with probabilities depending on the spin states of adjacent vertices. There exist two main Potts models. In the *ferromagnetic Potts model*, the state of a spin is attracted to the spin states of adjacent vertices (equilibria correspond to

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monochromatic graph colourings). This model is well-understood. In the *antiferromagnetic Potts model*, the state of a spin is repelled by the spin states of adjacent vertices, which means that every spin "tries" to achieve a state that is distinct from its neighbours. The behaviour of the antiferromagnetic Potts model remains elusive even for 2-dimensional models.

For many statistical mechanics systems for which we do not know exact solutions, (Markov chain) Monte Carlo simulations are valuable tools. For the antiferromagnetic Potts model, the Wang-Swendsen-Kotecký (WSK) non-local cluster dynamics [116, 117] is one of the most popular. In order to be *valid* (in other words, to work properly), Monte Carlo simulations require *ergodicity*, which means that there must be a positive probability of transforming each configuration into any other. Even if this property directly applies when the temperature of the system is positive, this condition does not necessarily hold at zero temperature.

In this latter case, the spin function corresponds to a proper q-colouring of G and the ergodicity of the Markov chain holds if and only if the set of q-colourings is a Kempe class. Since this property does not hold in general, the statistical mechanics community has studied the ergodicity of the Markov chain on special graphs, especially highly structured graphs that can be embedded on surfaces. Amongst them, *triangular lattices* and *kagomé lattices* have received considerable attention. A triangular lattice is a 6-regular graph embeddable on the torus in which every face is a triangle, and a kagomé lattice is a 4-regular graph embeddable on the torus in which every face is a triangle or a hexagon such that every edge of the lattice belongs to exactly one triangle and one hexagon (see Figure 3.6. In the following theorem we summarise what is currently known about the ergodicity of the WSK algorithm on these lattices including new results implied by Theorem 3.14.

Theorem 3.16. *The WSK algorithm for q-colourings of the triangular lattice is valid if $q \geq 6$ and is not valid if $q \leq 4$.*
The WSK algorithm for q-colourings of the kagomé lattice is valid if $q \geq 4$ and not valid if $q \leq 3$.

Proof. For the triangular lattice, Mohar and Salas showed in [98] that the chain is not ergodic when $q \leq 4$. Theorem 3.14 ensures that when $q = 6$, as the triangular

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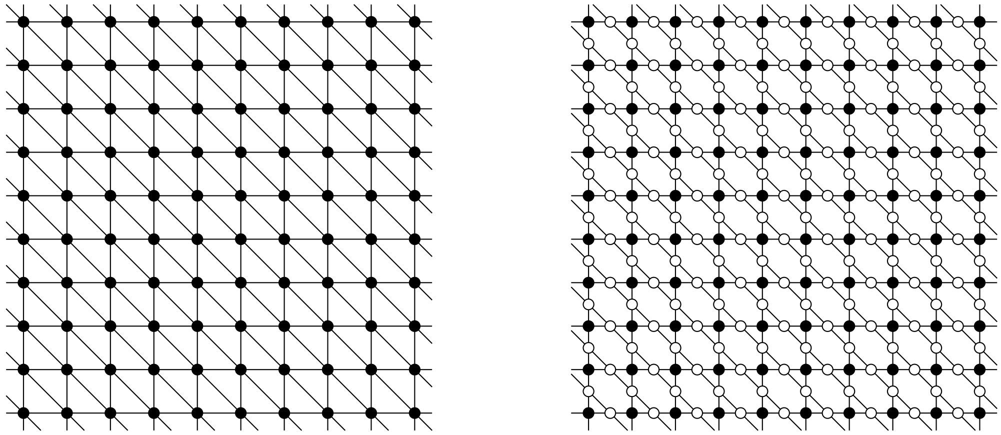

Figure 3.6: Portions of a triangular lattice and a Kagomé lattice

lattice is 6-regular, the Markov chain is ergodic and then the WSK algorithm is valid. For larger values of q, a result of Las Vergnas and Meyniel [114] ensures that the chain is also ergodic.

For the kagomé lattice, Mohar and Salas proved in [99] that the chain is not ergodic when $q \leq 3$. Theorem 3.14 ensures that when $q = 4$, as the kagomé lattice is 4-regular, the Markov chain is ergodic and then the WSK algorithm is valid. For larger values of q, a result of Las Vergnas and Meyniel [114] ensures that the chain is also ergodic. ■

We observe that this leaves the single open case of a triangular lattice with $q = 5$.

3.3.2 Preliminaries

Let d be a positive integer. Then a *d-elimination ordering* of the vertices of G is an ordering such that each vertex is adjacent to at most d vertices later in the ordering. We say that the ordering ends in S if the vertices of S are later in the ordering than all other vertices. Recall that a graph is *d-degenerate* if there is a d-elimination ordering of its vertices. From these definitions we immediately obtain:

Lemma 3.17. *Let d be a positive integer. Let G be a graph, and let S be a subset of the vertices of G. Then if G admits a d-elimination ordering that ends in S, then any $(d+1)$-colouring of $G[S]$ can be extended to G.*

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Let us refine this in a way that will prove useful.

Lemma 3.18. *Let k be a positive integer. Let $G = (V, E)$ be a graph, and let $S \subseteq V$, $|S| \leq k$, be a subset of the vertices. Suppose that $G[V \setminus S]$ is connected, that the vertices of $V \setminus S$ each have degree at most k in G and there is a vertex $x \in V \setminus S$ of degree at most $k-1$ in G. Then any k-colouring of $G[S]$ can be extended to G.*

Proof. Let the vertices of $V \setminus S$ be ordered according to the order in which they are found by a breadth-first search from x. Append the vertices of S to this ordering. This is certainly a $(k-1)$-elimination ordering of G since x has at most $k-1$ neighbours in total, every other vertex in $V \setminus S$ has at most k neighbours but at least one — the vertex from which is was discovered during the breadth-first search — is earlier in the ordering, and each vertex of S is followed in the ordering only by other vertices of S of which there are at most $k-1$. So, by Lemma 3.17 with $d = k-1$, the k-colouring of S can be extended to G. ∎

We need some known results.

Lemma 3.19 ([114])**.** *Let k be a positive integer. Let G_1, G_2 be two graphs such that $G_1 \cap G_2$ is complete. If both $C^k_{G_1}$ and $C^k_{G_2}$ are Kempe classes then $C^k_{G_1 \cup G_2}$ is a Kempe class.*

Recall that we *identify* two non-adjacent vertices u and v in a graph G if we replace them by a new vertex adjacent to all neighbours of u and v in G. The graph obtained is denoted G_{u+v}. In the proof of Theorem 3.14 we will often think about G_{u+v} when reasoning about the colourings of G. Let $C^k_G(u, v)$ denote the colourings of G for which u and v are coloured alike. We note that there is an obvious bijection between $C^k_G(u, v)$ and $C^k_{G_{u+v}}$.
The proof of Lemma 3.11 first establishes the following statement which it is useful to state explicitly.

Lemma 3.20 ([53])**.** *Let $k \geq 3$ be a positive integer. Let G be a 3-connected graph of maximum degree k. Let u and v be non-adjacent vertices of G with a common neighbour. Then G_{u+v} is $(k-1)$-degenerate.*

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A list assignment of a graph $G = (V, E)$ is a function L with domain V such that, for each vertex $u \in V$, $L(u)$ is a set of colours. We say that G is *L-colourable* if there is a colouring of G where every vertex u is coloured with a colour of $L(u)$, and G is *degree-choosable* if it is L-colourable for any list assignment L where, for each vertex u in G, the length of the list $L(u)$ is equal to the degree of u. The *blocks* of a graph are its maximal 2-connected subgraphs. The following well-known fact is a special case of the characterization of degree-choosable connected graphs of Borodin [18] and Erdős et al. [48].

Lemma 3.21 ([18, 48]). *Let G be a connected graph. Then G is degree-choosable unless each block of G is a complete graph or an odd cycle.*

More definitions. Given two sets S_1 and S_2 of vertices of G, we say that S_1 *dominates* S_2 if every vertex in S_2 is adjacent to at least one vertex in S_1. Additionally, S_1 *weakly dominates* S_2 if every vertex in S_2 is adjacent to exactly one vertex in S_1.

3.3.3 The Proof of Theorem 3.14

We must show that, for $k \geq 4$, if G is a connected non-complete k-regular graph, then the set of k-colourings of G is a Kempe class. In Propositions 3.22, 3.28 and 3.29, we show that this claim holds, respectively, whenever G is not 3-connected, 3-connected with diameter at least 3 and with diameter exactly 2. It is clear that taken together the propositions imply Theorem 3.14.

Graphs that are not 3-connected

We first prove that the Theorem 3.14 holds when G is not 3-connected.

Proposition 3.22. *Let $k \geq 4$ be a positive integer. Let G be a connected k-regular graph that is not 3-connected. Then C_G^k is a Kempe class.*

Proof. Let S be a minimal vertex cut of G that separates a connected component C_1 of $G - S$ from the rest of the graph C_2. Let $G_1 = G[C_1 \cup S]$ and $G_2 = G[C_2 \cup S]$. Note that both G_1 and G_2 are $(k-1)$-degenerate. Thus $C_{G_1}^k$ and $C_{G_2}^k$ are Kempe classes by Lemma 3.7.

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If $G[S]$ is a clique, then, by Lemma 3.19, C_G^k is a Kempe class.

As G is not 3-connected, $|S| \leq 2$. So if $G[S]$ is not a clique, then $S = \{x, y\}$ where x and y are a pair of non-adjacent vertices. We can assume that one vertex of S has more than one neighbour in G_1 and the other has more than one neighbour in G_2; suppose instead that, for example, x and y both have only one neighbour in G_1 and note that we can, in this case, let S be the cut of size 2 containing y and the unique neighbour of x in G_1 and now S does have the desired property. So we can assume, without loss of generality, that x has at least two neighbours in G_1, and y has at least two neighbours in G_2.

Let G_1', G_2' and G' be the graphs obtained from, respectively, G_1, G_2 and G by adding the edge xy. As x has degree at least 2 in G_1, it has degree at most $k-2$ in G_2 and thus degree at most $k-1$ in G_2'. Similarly y has degree at most $k-1$ in G_1'. Hence G_1' and G_2' are $(k-1)$-degenerate and, by Lemma 3.7, $C_{G_1'}^k$ and $C_{G_2'}^k$ are Kempe classes. By Lemma 3.19, $C_{G'}^k$ is a Kempe class.

So the set of k-colourings of G in which x and y have distinct colours are all Kempe equivalent (since this is the set of k-colourings of G'). To prove that C_G^k is a Kempe class, it remains to show that every k-colouring α of G such that $\alpha(x) = \alpha(y)$ is Kempe equivalent to a k-colouring where x and y are coloured differently. We will describe how to find a series of Kempe changes that, starting from α, give us a colouring in which x and y are not coloured alike.

We can assume that $\alpha(x) = \alpha(y) = 1$. If, for either x or y, there is a colour that does not appear on any vertex in its neighbourhood, then we can apply a trivial Kempe change to obtain the required colouring. So we assume that, under α, for each of x and y, there is a neighbour of each colour and so exactly one colour appears on two neighbours. We consider two cases.

Case 1: *Either x or y has at least two neighbours in each of G_1 and G_2.*

Let us assume that it is x that has two neighbours in both G_1 and G_2. There exist two colours — let us say 2 and 3 — such that no neighbour of x in G_1 is coloured 3 and no neighbour of x in G_2 is coloured 2. Consider the $(2, 3)$-components of G that include the neighbours of x coloured 2. Since they are included in G_1, they do not contain any neighbour of x coloured 3. So in the colouring obtained by a Kempe

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change of these components, the vertex x has no neighbour coloured 2. Thus by one further trivial Kempe change of x, the required colouring is obtained.

Case 2: *Neither x nor y has at least two neighbours in each of G_1 and G_2.*

We can assume that x has exactly one neighbour w in G_2, and y has exactly one neighbour z in G_1 and that $\alpha(w) = 2$. If $\alpha(z) \neq 2$, then consider the $(2, \alpha(z))$-component that contains z. From the Kempe change of this component (which does not contain x, y or w), we obtain a colouring where z is coloured 2. Thus we can as well assume that $\alpha(z) = 2$. Consider the $(1, 3)$-component that contains x. As x has no neighbour coloured 3 in G_2 and y has no neighbour coloured 3 in G_1, this component does not contain y. Thus from the Kempe change of this component we obtain the required colouring. ■

3-connected graphs with diameter at least 3

We present a number of lemmas that will allow us to show that Theorem 3.14 is true for 3-connected graphs with diameter at least 3.

If two neighbours t_1 and t_2 of a vertex u are not adjacent, we say that (t_1, t_2) is an *eligible* pair of neighbours of u. Let $P(u)$ denote the set of eligible pairs of neighbours of u. We observe that in a regular connected non-complete graph, every vertex has an eligible pair of neighbours.

The next lemma follows from Lemma 3.11. (In fact, it is just a special case of Lemma 3.11, but it is helpful to have it as a separate statement.)

Lemma 3.23. *Let k be a positive integer. Let G be a 3-connected k-regular graph G. Let u be a vertex in G and let (t_1, t_2) be an eligible pair in $P(u)$. Then $C_G^k(t_1, t_2)$ is a Kempe class.*

It is worth noting that as $G_{t_1+t_2}$ is $(k-1)$-degenerate it has a k-colouring so $C_G^k(t_1, t_2)$ is non-empty.

Lemma 3.24. *Let $k \geq 4$ be a positive integer. Let G be a 3-connected k-regular graph. Let u and v be two vertices of G and let (w_1, w_2) be an eligible pair in $P(v)$. If, for every eligible pair (t_1, t_2) in $P(u)$, there is a k-colouring of G such that t_1 and t_2 are coloured alike and w_1 and w_2 are coloured alike, then C_G^k is a Kempe class.*

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Proof. In a k-colouring of G at most $k-1$ colours appear on the neighbours of u. Thus at least two of its neighbours, which must be an eligible pair, are coloured alike. That is, for every colouring α of G, there is an eligible pair (t_1, t_2) in $P(u)$ such that α belongs to $C_G^k(t_1, t_2)$. So

$$C_G^k = \bigcup_{(t_1,t_2)\in P(u)} C_G^k(t_1, t_2),$$

and, as each $C_G^k(t_1, t_2)$ is, by Lemma 3.23, a Kempe class, we have that C_G^k is a Kempe class if it contains a subset that is a Kempe class and intersects $C_G^k(t_1, t_2)$ for each $(t_1, t_2) \in P(u)$. The premise of the lemma is that $C_G^k(w_1, w_2)$ intersects each $C_G^k(t_1, t_2)$ and it is, by Lemma 3.23, a Kempe class. ■

So Lemma 3.24 suggests an approach to proving that Theorem 3.14 holds for 3-connected graphs. We note first that it will be easier to apply if we know that G has diameter 3 since then we can choose u and v such that their eligible pairs of neighbours are distinct. We just need to prove that we can find the types of k-colourings that the premise of the lemma requires. To do this we need a number of rather technical lemmas.

Lemma 3.25. *Let $k \geq 4$ be a positive integer. Let G be a k-regular 3-connected graph with a vertex cut S of size 3 such that one connected component C of $G - S$ is a clique on k vertices. If S weakly dominates C, then C_G^k is a Kempe class.*

Proof. As C has at least four vertices each adjacent to exactly one of the three vertices of S, we can assume that there is a vertex in S with at least two neighbours in C. Let this vertex be u. Let w_1 be a neighbour of u in C. Let w_2 be a neighbour of u not in C. (If u does not have such neighbours, then $S \setminus \{u\}$ is a vertex cut and G is not 3-connected.)

By Lemma 3.23, $C_G^k(w_1, w_2)$ is a Kempe class. Let α be a k-colouring of G. The lemma follows if we can show that α is Kempe equivalent to a colouring in $C_G^k(w_1, w_2)$; that is, if by performing a number of Kempe changes we can reach a colouring where w_1 and w_2 are coloured alike.

Let us assume that $\alpha(w_1) = 1$. If $\alpha(w_2) = 1$, we are done so assume that $\alpha(w_2) = 2$. Let w_3 be the vertex in C for which $\alpha(w_3) = 2$ (as C is a clique on k vertices every colour appears on exactly one vertex).

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If w_3 is a neighbour of u, then $\{w_1, w_3\}$ is a Kempe chain and by a single Kempe change we obtain the required colouring. Otherwise suppose that the neighbour of w_3 in S is $v \neq u$. As u has at least two (distinctly coloured) neighbours in C, we can assume there is a neighbour w_4 of u in C such that $\alpha(w_4) \neq \alpha(v)$ (possibly $w_4 = w_1$). Then $\{w_3, w_4\}$ is a Kempe chain. If we exchange the colours of this chain, then either $w_4 = w_1$ and we are done or, as before, we have two neighbours of u coloured 1 and 2 which form a Kempe chain and one more Kempe change is needed to obtain the required colouring. ■

At various points in the proofs of the following lemmas we will have defined a graph G with vertices u and v and eligible pairs $(t_1, t_2) \in P(u)$ and $(w_1, w_2) \in P(v)$. Whenever this is the case we will use the following definitions. Let G^+ be the graph obtained from G by identifying t_1 and t_2 and then identifying w_1 and w_2, and label the two vertices created t and w respectively. Let G^- be the graph obtained from G^+ by deleting t and w (so G^- is the graph obtained from G by deleting t_1, t_2, w_1 and w_2).

Lemma 3.26. *Let $k \geq 4$ be a positive integer. Let G be a 3-connected k-regular graph. Let u and v be two vertices of G and let (w_1, w_2) be an eligible pair in $P(v)$ neither of which is adjacent to u. Suppose that C_G^k is not a Kempe class. Then there is an eligible pair (t_1, t_2) in $P(u)$, such that G contains an induced subgraph weakly dominated by both $\{t_1, t_2\}$ and $\{w_1, w_2\}$ that is isomorphic to K_{k-1}.*

Proof. As C_G^k is not a Kempe class, we know, by Lemma 3.24, we can choose as (t_1, t_2) an eligible pair in $P(u)$ such that there is no k-colouring of G such that t_1 and t_2 are coloured alike and w_1 and w_2 are coloured alike. We note that t_1, t_2, w_1 and w_2 are distinct as the latter two are not adjacent to u. So here G^+ is well-defined and, by our choice of t_1 and t_2, does not have a k-colouring. To prove the lemma, we attempt to construct a k-colouring of G^+ and use the fact that we know that we cannot succeed to lead us to the conclusion.

For a component C of G^-, let G_C^* be $G^+[C \cup \{t, w\}]$. For each C, we shall show that one of the following holds:

(1) the structure of G_C^* implies that G^+ has a k-colouring, or

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(2) there is a k-colouring of G^*_C where t and w are coloured 1 and 2 respectively, or

(3) $G[C]$ contains an induced subgraph weakly dominated, in G, by both $\{t_1, t_2\}$ and $\{w_1, w_2\}$ that is isomorphic to K_{k-1}.

By the assumption that G^+ has no k-colouring, there cannot be any component that satisfies (1) and it cannot be the case that every component satisfies (2). Thus there must be at least one component that satisfies (3) and the lemma follows.

Case 1: There is a vertex x in C that has degree less than k in G^*_C.

We can find a k-colouring of G^*_C with t and w coloured with 1 and 2 by applying Lemma 3.18 to G^*_C and x with $S = \{t, w\}$. So C satisfies (2).

Case 2: Every vertex in C has degree k in G^*_C and $G[C]$ is degree-chooseable.

We create a list assignment L for $G[C]$. For each vertex x in C, let

$$L(x) = \begin{cases} \{1, \ldots, k\} & \text{if } x \text{ is not adjacent to } t \text{ or } w, \\ \{2, \ldots, k\} & \text{if } x \text{ is adjacent to } t \text{ but not } w, \\ \{1, 3 \ldots, k\} & \text{if } x \text{ is adjacent to } w \text{ but not } t, \\ \{3 \ldots, k\} & \text{if } x \text{ is adjacent to both } t \text{ and } w. \end{cases}$$

Note that $|L(x)|$ is equal to the degree of x in $G[C]$ since it is $k - |N_{G^+}(x) \cap \{t, w\}|$. As $G[C]$ is degree-chooseable, there is a colouring of $G[C]$ that respects L and as $1 \notin L(x)$ if x is adjacent to t and $2 \notin L(x)$ if x is adjacent to w this provides a k-colouring of G^*_C when t and w are coloured 1 and 2. Thus C satisfies (2).

Case 3: Every vertex in C has degree k in G^*_C and $G[C]$ is not degree-chooseable.

By Lemma 3.21, each block of $G[C]$ is a either a clique or an odd cycle. For an end block B of $G[C]$, let B^- be the vertices of B that are not a cutvertex in $G[C]$ (so B^- contains one fewer vertex than B unless $G[C]$ contains only one block and then $B^- = B$). The degree of each vertex of B^- in G^*_C is k and this is the sum of the number of neighbours it has in C and the number of neighbours it has in $\{t, w\}$. As the former is the same for each vertex (as they belong to just one block that is a cycle or a clique), the latter must also be the same for each vertex. So let $d_B \in \{0, 1, 2\}$ be the number of neighbours in $\{t, w\}$ of each vertex of B^-.

Case 3.1: There is an end block B of C with $d_B = 0$.

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This implies that each vertex of B^- is joined to k vertices in C which, as $k \geq 4$, implies that B is a clique rather than a cycle and so B is isomorphic to K_{k+1} contradicting that G is connected and non-complete.

Case 3.2: There is an end block B of C with $d_B = 1$.

Note that B must be a clique as if it were an odd cycle the degree of each vertex of B^- in G^*_C would be $3 \neq k$.

Suppose every vertex in B^- is adjacent to t (the case where they are all adjacent to w is equivalent). We cannot have $B = B^-$ since then t is a cutvertex and so $\{t_1, t_2\}$ is a cutset in G contradicting that it is 3-connected. So let x be the cutvertex of $G[C]$ in B. Then x has exactly one neighbour s in $C \setminus B^-$. Thus $\{s, t_1, t_2\}$ is a vertex cut of G that weakly dominates B which is a clique on k vertices. Therefore C^k_G is a Kempe class by Lemma 3.25; a contradiction.

So there must be vertices y and z in B^- such that y is adjacent to t (but not w) and z is adjacent to w (but not t). We show that we can colour t and w with 1 and 2 and extend this to a k-colouring of G^*_C. First colour z with 1. Then apply Lemma 3.18 to $G^*_C \setminus \{y\}$ with $S = \{t, w, z\}$ and x being a vertex other than y and z in B^- (if B^- does not contain three vertices, then the degree of y and z in G^*_C is at most $3 < k$). Finally colour y, which is possible as two of its neighbours are coloured alike. Thus C satisfies (2).

For the remaining cases, we will need the following claim.

Claim 1. *If u and v are not in C, then*

A. *each of t and w is adjacent to at most $2k-2$ vertices in C,*
B. *one of t and w is adjacent to at most $2k-3$ vertices in C,*
C. *if each of t and w has at least $2k-3$ neighbours in C, then t is not adjacent to w, and*
D. *if the sum of the number of neighbours of t and w in C is at least $4k-6$, then $G^+[V \setminus C]$ has a k-colouring in which t and w are coloured alike.*

We note that this claim can be applied within Case 3 as we know that every vertex in C has degree k in G^*_C and u and v have degree less than k since a pair of neighbours — t_1 and t_2 or w_1 or w_2 — were identified when G^+ was formed from G. We prove

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each part of the claim (we give a proof only for the statement about t when the argument for w is equivalent). We keep in mind that for each edge incident with t in G^+ there is a corresponding edge or edges incident with t_1 or t_2 in G.

A. The total number of edges incident with t_1 and t_2 in G is $2k$, but two of these are incident with u which is not in C.

B. If t and w both have $2k-2$ neighbours in C, then in G, t_1, t_2, w_1 and w_2 only have neighbours in $C \cup \{u, v\}$. Then $\{u, v\}$ is a cutset as it separates $C \cup \{t_1, t_2, w_1, w_2\}$ from the rest of G which is not empty as u has at least 4 neighbours and is not adjacent to any vertex in $C \cup \{w_1, w_2\}$. This contradicts that G is 3-connected.

C. If t and w both have $2k - 3$ neighbours in C and are adjacent, then, in G, t_1, t_2, w_1 and w_2 only have neighbours in $C \cup \{u, v, t_1, t_2, w_1, w_2\}$, and, as in the previous part, this implies that $\{u, v\}$ is a cutset.

D. We can say that t and w are not adjacent: either one of t and w has $2k - 2$ neighbours in C so its only other neighbour is either u or v, or they both have $2k - 3$ neighbours in C and so we can apply the previous part of the claim. In G, there are at least $4k - 6$ edges from $\{t_1, t_2, w_1, w_2\}$ to the vertices of C so at most 6 other incident edges. And, as t_1 and t_2 are both adjacent to u and w_1 and w_2 are both adjacent to v, in $G^+[V \setminus C]$ the sum of the degrees of t and w is at most 4. Let $G^\dagger$ be the graph formed from $G[V \setminus C]$ by identifying t and w to form a new vertex with degree at most 4. Thus every vertex in $G^\dagger$ has degree at most k and the graph is not isomorphic to K_{k+1} (since u, for example, has degree less than k) so, by Brooks' Theorem, $G^\dagger$ has a k-colouring. From this colouring, we can obtain a colouring of $G^+[V \setminus C]$ in which t and w are coloured alike. This completes the proof of the claim.

Case 3.3: For every end block B of C, $d_B = 2$, and there is one end block B_1 that is not a clique.

So B_1 is an odd cycle on at least five vertices. In G^*_C, each vertex of B_1^- has degree k and is adjacent to two vertices in B_1 and t and w so $k = 4$. If either B_1 has more than five vertices or C has more than one end block, then there are at least six vertices in end blocks that are not cutvertices and so are adjacent to both t

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and w which therefore both have at least $6 = 2k-2$ neighbours in C contradicting Claim 1.B. So $C = B_1$ is a 5-cycle and the sum of the number of neighbours of t and w in C is $10 = 4k-6$ so, by Claim 1.D, $G^+[V \setminus C]$ has a 4-colouring in which t and w are coloured alike. We can extend this colouring to the whole of G^+ by using the other 3 colours on B. So C satisfies (1).

Case 3.4: For every end block B of C, $d_B = 2$ and B is a clique.

Notice that each end block is isomorphic to K_{k-1}. If there is only one end block, then, as it is weakly dominated by $\{t_1, t_2\}$ and $\{w_1, w_2\}$ in G, C satisfies (3). If there are at least three end blocks, then there are $3(k-2)$ vertices in C adjacent to both t and w. As, for $k \geq 4$, we have $3k-6 \geq 2k-2$, this contradicts Claim 1.B.

So we can assume that C has exactly two end blocks each isomorphic to K_{k-1}. Note that an "intermediate" block B of C that is a clique on more than two vertices has vertices (the ones that are not cutvertices in $G[C]$) whose k neighbours are each either in B or in $\{t, w\}$. In fact, at least one neighbour must be in $\{t, w\}$ else B is isomorphic to K_{k+1} and not connected to the rest of G. Therefore B is isomorphic to either K_{k-1} or K_k.

Case 3.4.1: $k \geq 5$.

No block is an odd cycle (since the vertices that are not cutvertices in the cycle would have degree at most 4 in G^*_C). So the blocks of C are each isomorphic to K_2, K_{k-1} or K_k and for each cutvertex one of the two blocks it belongs to must be K_2 else it would have degree at least $2(k-2) > k$. Thus the cutvertex of each end block is also adjacent to one of t and w so there are $4k-6$ edges from t and w to vertices of the two end blocks. If there is an intermediate block that is isomorphic to K_{k-1} or K_k, then it contains at least two vertices that are not cutvertices and these are also joined to at least one of t and w. So the sum of the number of neighbours of t and w in C is at least $4k-4$; a contradiction to the first two parts of Claim 1. Therefore the only intermediate block is K_2 and there is exactly one of these (if there are none the two end blocks intersect and the cutvertex has degree more than k; if there is more than 1, there are vertices that in $G[C]$ have degree 2 so have degree at most 4 in G^*_C). So $G[C]$ contains two disjoint cliques each isomorphic to K_{k-1} joined by a single edge. Thus the sum of the number of neighbours of t and w in C is exactly

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$4k-6$ and we can assume, by Claim 1.D, that $G^+[V \setminus C]$ has a k-colouring in which t and w are coloured alike. This can be extended to a colouring of G^+ as $G[C]$ is easily seen to be $(k-1)$-colourable. So C satisfies (1).

Case 3.4.2: $k = 4$.

Let the two end blocks be B_1 and B_2 (both are isomorphic to K_3). If they intersect in a vertex, then we can colour t and w with 1 and 2, colour the vertex in both B_1 and B_2 with 1 and the other vertices with 3 and 4. So C satisfies (2).

For the remaining cases, we note that Claim 1.D says that if there are at least 10 edges joining t and w to C we can assume they are coloured alike in a 4-colouring of $G^+[V \setminus C]$. And Claim 1.A and B say that there cannot be more than 11 edges from t and w to C.

If $G[C]$ is B_1 and B_2 plus an edge between them, then there are 10 edges from t and w to C and clearly $G[C]$ is 3-colourable so C satisfies (1).

Suppose that $G[C]$ contains more blocks than B_1 and B_2 and an additional K_2. If C does not contain a K_4, then either there is a block isomorphic to K_3 or a longer odd cycle that contains a vertex x that is not a cutvertex, or there is a cutvertex x that belongs to two blocks both isomorphic to K_2. In both cases x must be joined to both t and w which are therefore again joined by at least 10 edges to C and as there is no K_4, $G[C]$ is 3-colourable and C again satisfies (1).

If C does contain a K_4, then the two vertices that are not cutvertices are both incident to one of t and w. And the cutvertices in B_1 and B_2 are each either adjacent to one of t or w or belong to a K_3 or a longer odd cycle that contains a vertex adjacent to both t and w. In any case, t and w are incident to at least 12 edges joining them to C and we have a contradiction. ■

Lemma 3.27. *Let $k \geq 4$ be a positive integer. Let G be a 3-connected k-regular graph. Let u and v be two vertices of G that are not adjacent. Let (w_1, w_2) be an eligible pair in $P(v)$ neither of which is adjacent to u. Then C_G^k is a Kempe class.*

Proof. If C_G^k is not a Kempe class, then, by Lemma 3.26, there is an eligible pair (t_1, t_2) in $P(u)$ such that G contains an induced subgraph isomorphic to K_{k-1} that is weakly dominated by both $\{t_1, t_2\}$ and $\{w_1, w_2\}$. Let C be the vertex set of this

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induced subgraph and note that each vertex in C is adjacent to the other $k-2$ vertices of C and to one of $\{t_1, t_2\}$ and one of $\{w_1, w_2\}$ and so is not adjacent to u or v (neither of which can be in C as they are each adjacent to both of the vertices in either $\{t_1, t_2\}$ or $\{w_1, w_2\}$). We can assume that each of $\{t_1, t_2, w_1, w_2\}$ is adjacent to at least one vertex in C: if fewer than three of them have a neighbour in C, then G is not 3-connected, and if exactly one of them, say t_1, has no neighbour in C, then, since $C \cup t_2$ would induce a clique on k vertices that is weakly dominated by $\{u, w_1, w_2\}$ (every vertex in C is adjacent to one of $\{w_1, w_2\}$, but not to u, and t_2 is adjacent to u, but, considering its degree, not to either of $\{w_1, w_2\}$) and Lemma 3.25 is contradicted.

Assume, without loss of generality, that w_1 has at least as many neighbours in C as w_2. Let x be a neighbour of w_1 in C and assume, without loss of generality, that x is also a neighbour of t_1. Then (x, v) is an eligible pair in $P(w_1)$. We apply Lemma 3.26 to u, w_1 and (x, v). So, under the assumption that C_G^k is not a Kempe class, there is a pair (t_3, t_4) (not necessarily distinct from (t_1, t_2)) of eligible neighbours in $P(u)$ such that G contains an induced subgraph isomorphic to K_{k-1} that is weakly dominated by both $\{t_3, t_4\}$ and $\{x, v\}$. Let C' be the vertex set of this induced subgraph and, arguing as we did for C, we can assume that each of $\{t_3, t_4, x, v\}$ is adjacent to C'.

Suppose that neither t_1 nor t_2 belongs to C'. The k neighbours of x are $C \setminus \{x\} \cup \{t_1, w_1\}$ and we know at least one of these vertices is in C'. By definition it is not w_1 and by assumption it is not t_1 so there is a vertex $y \neq x$ that belongs to both C and C'. As C' induces a clique, the other $k-2$ vertices of C' are neighbours of y. But as none of $\{t_1, t_2, w_1, x\}$ are in C', we must have that C' is $C \setminus \{x\} \cup \{w_2\}$. So w_2 is adjacent to every vertex of C except x. By our assumption that w_1 has at least as many neighbours as w_2 in C, we have that C has only two vertices and so $k = 3$. This contradiction tells us that, in fact, at least one of t_1 and t_2 belongs to C'; let us assume it is t_1.

So t_1 has $k-2$ neighbours in C'. It has two more neighbours: we know it must be adjacent to one of $\{v, x\}$ by the definition of C', and we know that it is also adjacent to u. But neither of t_3 and t_4 belongs to $C' \cup \{u, v, x\}$ so t_1 is not adjacent

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to either of them. This contradicts the definition of C' and completes the proof. ■

We can now conclude this subsection on graphs of diameter at least 3.

Proposition 3.28. *Let $k \geq 4$ be a positive integer. Let G be a 3-connected k-regular graph with diameter at least 3. Then $C_k(G)$ is a Kempe class.*

Proof. Let u and v be two vertices in G at distance at least 3. Then every neighbour of v is not adjacent to u and the result follows from Lemma 3.27. ■

3-connected graphs with diameter 2

To complete the proof of Theorem 3.14, it only remains to consider 3-connected graphs of diameter 2.

First two definitions. For a vertex v in a graph G, we denote by $N(v)$ the *neighbourhood* of v, that is, the set of vertices adjacent to v. The *second neighbourhood* of v is the subgraph of G induced by the set of vertices at distance 2 from v in G.

Proposition 3.29. *Let $k \geq 4$ be a positive integer. Let G be a 3-connected k-regular graph of diameter 2. Then $C_k(G)$ is a Kempe class.*

Proof. If the second neighbourhood of a vertex v contains an induced path on three vertices, then the proposition follows immediately from Lemma 3.27. Therefore we can assume that the second neighbourhood of each vertex is a disjoint union of cliques.

Assume that there is a vertex v whose second neighbourhood contains two cliques C_1 and C_2. Let x and y be vertices of C_1 and C_2 respectively. If x is adjacent to a neighbour z of v that is not adjacent to y, then the second neighbourhood of y contains an induced path on v, z and x and, again, we are done by Lemma 3.27. Thus, by symmetry, the intersections of each of the neighbourhoods of x and y with $N(v)$ are the same and, repeating the argument, we must have that every vertex of C_1 and C_2 has the same set of neighbours within $N(v)$. Let α be a k-colouring of G. Suppose that $\alpha(x) = 1$ and $\alpha(y) = 2$. Note that the $(1, 2)$-component that contains x contains only vertices of C_1. Exchange the colours on this $(1, 2)$-component and let β be the resulting colouring. So $\beta(x) = \beta(y) = 2$. Thus from any k-colouring,

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we can obtain by a single Kempe change a colouring in $C^k_G(x, y)$. The proposition follows from Lemma 3.11.

Therefore we can assume that the second neighbourhood of each vertex is a clique. Let α and β be two k-colourings of G. Let v be a vertex and let us denote by C the second neighbourhood of v. Up to a Kempe change, we can assume that $\alpha(v) = \beta(v) = 1$. To complete the proof, we assume that α and β are not Kempe equivalent and show that this leads to a contradiction.

Claim 2. *Neither α nor β is Kempe equivalent to a colouring γ such that $\gamma(v) = 1$ and the colour 1 is not used in C.*

Suppose that there is such a colouring γ that is Kempe equivalent to, say, α. Let x be the vertex in C with $\beta(x) = 1$ if such a vertex exists; otherwise let x be any vertex in C. In γ, v is the only vertex in G coloured 1 (since certainly there is no vertex in $N(v)$ coloured 1) so we can apply a trivial Kempe change to x from γ to obtain a colouring γ' where $\gamma'(x) = 1$. If no vertex in β is coloured 1, then we can use the same argument; that is, apply a trivial Kempe change to x to obtain a colouring where x is coloured 1. So we may as well assume that $\beta(x) = 1$, and thus, as v and x are coloured 1 in both γ' and β, we have, by Lemma 3.11 that γ' and β, and so also α and β, are Kempe equivalent; a contradiction that proves the claim.

One thing that Claim 2 tells us is that α and β are colourings where the colour 1 is used on C. So let u and w be vertices in C such that $\alpha(u) = 1$ and $\beta(w) = 1$. If $u = w$, then Lemma 3.11 implies that α and β are Kempe equivalent. So, by assumption, we have $u \neq w$.

One more definition: given a colouring γ, a vertex x is *locked* if all the colours distinct from $\gamma(x)$ appear in its neighbourhood. Notice that if x is not locked, then we can apply a trivial Kempe change to x from γ.

Claim 3. *Each vertex in $u \cup N(u) \setminus w$ is locked in α. Moreover, only colour $\alpha(w)$ appears twice in the neighbourhood of u.*

First consider the $(1, \alpha(w))$-component of α containing u and w. If this component does not contain v, then the Kempe change of this component from α gives us a colouring in which w and v are both coloured 1. By Lemma 3.11, this colouring is

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Kempe equivalent to β, a contradiction. Thus v must be in the $(1, \alpha(w))$-component. Since no other neighbour of w distinct from u is coloured 1 (every vertex in G is a neighbour of v or u), another neighbour y of u must be coloured with $\alpha(w)$. If u is not locked, then a trivial Kempe change of u gives us a colouring in which 1 is not used on C, contradicting Claim 2. Thus all the colours appear exactly once on the neighbourhood of u except colour $\alpha(w)$ which appears twice. If y is not locked, then a trivial Kempe change of y returns us to the case where the $(1, \alpha(w))$-component of α containing u and w does not contain v. And if a neighbour z of u not in $\{w, y\}$ is not locked, then a trivial Kempe change of z returns us to the case where u is not locked. The claim is proved.

Case 1: $|C| \geq 3$.

Let $z \in C \setminus \{u, w\}$. Clearly u is the unique neighbour of z coloured with 1 in α (since, again, every in G is a neighbour of v or u). Similarly w is the unique neighbour of z coloured with 1 in β. By Claim 3, z is the unique neighbour of u coloured $\alpha(z)$, and so $\{u, z\}$ is a Kempe chain in α. Similarly, noting that Claim 3 also holds for β with the roles of u and w interchanged, $\{w, z\}$ is a Kempe chain in β. By exchanging the colours on these Kempe chains we obtain two colourings where v and z are each coloured 1. Lemma 3.11 then implies that α and β are Kempe equivalent, a contradiction.

Case 2: $|C| = 2$.

So G contains v, its k neighbours, and u and w. Each of u and w are adjacent to all but one of the neighbours of v, so at least $k - 2$ of the neighbours of v are adjacent to both u and w; let this set of neighbours be denoted S. By Claim 3, in α a common neighbour z of u and v is coloured $\alpha(w)$, so it follows that S contains exactly $k - 2$ vertices as it cannot contain z. As each vertex of S is locked in α by Claim 3 and has two neighbours, u and v, coloured 1, they each have exactly one neighbour of each other colour. Thus as w and z are coloured alike and every vertex in S is adjacent to w, no vertex in S is adjacent to z. But then the only vertices that can be adjacent to z are u, v and the other neighbour of v that is not in S which contradicts that $k \geq 4$. This completes Case 2 and the proof of the proposition. ■

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3.4 Miscellaneous Results

Let $e = uv$ be an edge of G. The operation of removing the edge e and identifying its ends u, v into a new vertex is called an *edge contraction*. A graph G is *contractible* to a graph H if H can be obtained from G by successively contracting a subset of the edges of G. Consider the following conjecture of Las Vergnas and Meyniel [114]. The *Hadwiger index* of a graph G, denoted by $\eta(G)$, is the greatest integer h such that G is contractible to the complete graph K_h.

Conjecture 3.30 ([114])**.** *$C_q(G)$ is a Kempe class for all integers $q \geq \eta(G) + 1$.*

The same authors were able to settle some special cases of the conjecture.

Theorem 3.31 ([114])**.** *If $\eta(G) \leq 4$, then $C_q(G)$ is a Kempe class for all integers $q \geq \eta(G) + 1$.*

Our next result provides a partial answer for the case $\eta(G) = 5$. Denote by $\mathscr{H}_5$ the class of graphs contractible to K_5 such that $G - v$ is not contractible to K_5 for some vertex v in G. Let α be a colouring of a graph G and let H be a graph obtained from G by deleting a set S of vertices or identifying a pair x, y of vertices coloured alike under α into a new vertex z. The *restriction* of α to H, denoted α_H, is the colouring that agrees with α on $V(G) \setminus S$ and for which $\alpha_H(z) = \alpha(x)(= \alpha(y))$.

That we have $\chi(G) = 4$ whenever $\eta(G) = 4$ is well-known and follows from Wagner's Theorem and the Four Colour Theorem:

Theorem 3.32. *Let G be a graph that is not contractible to K_5. Then G has a 4-colouring.*

Proposition 3.33. *Let $G = (V, E) \in \mathscr{H}_5$. Then $C_6(G)$ is a Kempe class.*

Proof. By assumption, there is a vertex $v \in V(G)$ such that the graph $G' = G - v$ is not contractible to K_5. Let ψ be a 6-colouring of G and assume up to a single Kempe change that $\psi(v) = 6$. Define $H = G - U$ where $U = \{u \in V \mid \psi(u) = 6\}$.

Claim 1. *There exists a 5-colouring ψ' of G such that $\psi \sim_6 \psi'$.*

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Since $v \in U$, the graph H is not contractible to K_5. By Theorem 3.31, $C_5(H)$ constitutes a Kempe class. By Theorem 3.32, the restriction ψ_H of ψ to H is Kempe equivalent to a 4-colouring ϕ of H in $C_5(H)$. Therefore ψ is Kempe equivalent to a 6-colouring ψ' of G for which $\psi'_H = \phi$ and $\psi'(U) = \psi(U) = \{6\}$. The claim is proved.

Let α and β be two 6-colourings of G. To complete the proof we show that $\alpha \sim_6 \beta$. Up to a single Kempe change, we may assume that $\alpha(v) = \beta(v) = 6$. By Claim 1, there are 5-colourings α' and β' of G such that $\alpha \sim_6 \alpha'$ and $\beta \sim_6 \beta'$. Define α'' to be the 6-colouring obtained from α' by recolouring v to colour 6. Also define analogously the 6-colouring β''. Since $\eta(G') = 4$, it follows by Theorem 3.31 that $\alpha''_{G'} \sim_5 \beta''_{G'}$. Thus $\alpha'' \sim_6 \beta''$ implying $\alpha \sim_6 \gamma$ as required. ■

A graph is *planar* if it can be drawn in the plane such that no two of its edges cross. A graph is *apex* if it can be made planar by deleting a vertex. The following is an immediate consequence of Proposition 3.33.

Corollary 3.34. *If G is an apex graph, then $C_6(G)$ is a Kempe class.*

Proof. A graph is planar if and only if the graph is not contractible to K_5 and $K_{3,3}$. ■

We now give a short proof of a theorem of Meyniel [94] which states that all 5-colourings of a planar graph are Kempe equivalent. Our proof relies on the Four Colour Theorem in contrast to the proof given in [94]. We shall need the following simplified version of a theorem due to Thomassen [112].

Theorem 3.35 ([112])**.** *Let G be a plane graph with outer face C. There exists a partition of $\{V_1, V_2\}$ of $V(G)$ such that V_1 induces an independent set, V_2 induces a 3-degenerate graph and $V(C) \subseteq V_2$.*

Corollary 3.36. *Let G be a planar graph. For all vertices $v \in V(G)$ satisfying $deg(v) \geq 3$, there exists a partition of $\{V_1, V_2\}$ of $V(G)$ such that $V_1 \ni v$ induces an independent set and V_2 induces a 3-degenerate graph.*

Proof. Denote by $u_1, u_2, \ldots, u_k$ the neighbours of v, $k \geq 3$, and assume that they occur in this cyclic ordering in a plane embedding of G. Consider the graph G'

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obtained from G by deleting vertex v and adding the edges u_1u_k and, for $1 \leq i \leq k-1$, u_iu_{i+1} (deleting multiple edges if necessary). Notice that G' is planar and the cycle $C = u_1u_2 \ldots u_ku_1$ forms the boundary of a face in G'. Thus, there exists a plane embedding of G' with outer cycle C. Applying Theorem 3.35 to G' we obtain a partition $\{V_1, V_2\}$ of $V(G')$ such that V_1 induces an independent set, V_2 induces a 3-degenerate graph and $V(C) \subseteq V_2$. This implies the corollary. ■

We shall also need the following result which is an immediate corollary of either of the proofs of Lemma 3.7.

Corollary 3.37. *Let G be a graph and let v be a vertex of G with degree at most k. If $C_{k+1}(G-v)$ is a Kempe class, then $C_{k+1}(G)$ is a Kempe class.*

Theorem 3.38 (Meyniel [94])**.** *Let G be a planar graph. Then $C_5(G)$ is a Kempe class.*

Proof. We use induction on the number of vertices $n = |V(G)|$. If $n \leq 6$ then G is 4-degenerate so the result follows by Lemma 3.7. Suppose $n \geq 7$ and that the result is true for all graphs with less vertices. Let v be a vertex of degree at most 5 in G (such a vertex exists since planar graphs are 5-degenerate). We start with a claim.

Claim 2. *Let ψ be a 5-colouring of G. If $deg(v) = 5$, then there is a 5-colouring ψ' of G Kempe equivalent to ψ such that $\psi'(v) = 5$ and ψ'_{G-v} is a 4-colouring.*

Since $deg(v) = 5$, there exists two neighbours x, y of v such that $\psi(x) = \psi(y)$. Let G' be the graph obtained from G by identifying x, y. Note that the graph $G' - v$ is planar and so, by the induction hypothesis, its set of 5-colourings is a Kempe class. In particular, by the Four Colour Theorem, the restriction $\psi_{G'-v}$ of ψ to $G' - v$ is Kempe equivalent to a 4-colouring γ' of $G' - v$. Since v has degree 4 in G', it follows by Corollary 3.37 that $\psi_{G'}$ is Kempe equivalent to a 5-colouring γ'' of G' for which $\gamma''_{G'-v} = \gamma'$ is a 4-colouring. Applying Lemma 3.10 completes the proof of the claim.

Let α and β be 5-colourings of G. To prove the theorem we show that $\alpha \sim_5 \beta$. If $deg(v) \leq 4$, then, by the induction hypothesis, $C_5(G')$ is a Kempe class. By Corollary 3.37, $C_5(G)$ is also a Kempe class.

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Finally suppose $deg(v) = 5$. By Claim 2, α and β are Kempe equivalent to 5-colourings α' and β' respectively of G satisfying $\alpha'(v) = \beta'(v) = 5$ and α'_{G-v} and β'_{G-v} are 4-colourings. Now by Corollary 3.36 there is a partition $\{V_1, V_2\}$ of $V(G)$ such that $V_1 \ni v$ induces an independent set and V_2 induces a 3-degenerate graph. Denote by α'' and β'' the colourings obtained from α' and β' respectively by recolouring the vertices in $V_1 \setminus \{v\}$ to colour 5 (the colour that is not used in V_2 with either α' or β'). We focus on the colourings restricted to V_2 and as long as we do not use colour 5 we do not need to worry about adjacencies with V_1. By Lemma 3.7, $C_4(G[V_2])$ is a Kempe class. Therefore $\alpha'' \sim_5 \beta''$ implying $\alpha \sim_5 \beta$ as required. ∎

A graph is perfect if the chromatic number of every induced subgraph of the graph is equal to the order of the largest clique contained in that subgraph. Recall that Bertschi [5] showed that all k-colourings of a perfectly contractile graph are Kempe equivalent. A natural extension of this result is to show that all k-colourings of perfect graphs are Kempe equivalent. The following result of Meyniel [95] shows that this is not always possible.

Proposition 3.39 ([95]). *For every integer $q \geq 3$ there exists a perfect graph G with $\chi(G) = q - 1$ and a q-colouring of G that is not Kempe equivalent to some $\chi(G)$-colouring of G.*

We can, in fact, exclude one more case.

Proposition 3.40. *For every positive integer $q \geq 3$ there exists a perfect graph G such that $q = \chi(G)$ and $C_q(G)$ consists of at least two Kempe classes.*

Proof. It suffices to describe, for each $q \geq 3$, a graph F_q such that $\chi(F_q) = q$, F_q is perfect and $C_q(F_q)$ is two Kempe classes. If $q = 3$, then we let F_3 be the 3-prism. Otherwise, F_q is defined recursively as the graph obtained from the graph F_{q-1} and an independent set S_q of vertices of any given size by joining every vertex of F_{q-1} to every vertex of S_q. It is immediate that $\chi(F_q) = q$ and, as argued for the 3-prism, $C_q(F_q)$ is two Kempe classes. To see that F_q is perfect, notice that the graph Q obtained from any graph H by adding a vertex v to H and joining v to every vertex of H has the property that, for any induced subgraph Q' of Q, if $v \in V(Q')$ then

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$\chi(Q') = \chi(Q' - v) + 1$ and $\omega(Q') = \omega(Q' - v) + 1$. Therefore, by definition, Q is perfect if H is perfect. This implies F_q is perfect as required. ■

The remaining cases are still open.

Problem 3.41. *Let G be a perfect graph. Find all integers $q \geq \chi(G) + 2$ such that the q-colourings of G are Kempe equivalent in $C_q(G)$.*

We also present an easier problem in appearance.

Problem 3.42. *Let G be a perfect graph. Find the smallest integer $q \geq \chi(G) + 1$ such that the $\chi(G)$-colourings of G are Kempe equivalent in $C_q(G)$.*

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Chapter 4

Partitioning a Graph into Disjoint Cliques and a Triangle-free Graph

4.1 Introduction

Chudnovsky described in [29, 30] a complete characterization of bull-free graphs. Roughly, the structure theorem states that any bull-free graph can obtained from three basic graph classes $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ by using some prescribed set of graph operations. (For the purpose of this chapter we can omit the definitions of theses classes). Motivated by finding an algorithmic version of this result, the complexity of recognizing the class $\mathcal{T}_1$ was left as an open question by Thomassé, Trotignon and Vušković [111].

Problem 4.1 ([111])**.** *Let G be a graph. Determine the computational complexity of recognizing whether or not G belongs to $\mathcal{T}_1$.*

In an attempt to answer this question, we shall give for our purpose the following simplified (that is, more general) definition of the class.

Definition 4.2. A graph $G \in \mathcal{W}_1$ if the following holds:

(1) G is bull-free,

(2) there exists a partition $V = A \cup B$ such that $G[A]$ is P_3-free and $G[B]$ is K_3-free,

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(3) for every $v \in G[A]$ there exists a partition $N(v) \cap B = X_1 \cup X_2$ such that $G[X_i]$ $(i = 1, 2)$ is a stable set and every vertex of X_1 is adjacent to every vertex of X_2.

We consider a superclass of the class $\mathcal{W}_1$ defined as follows: a graph G is said to be *partitionable* or, unless stated otherwise, has a *partition* if there exists a partition $\{A, B\}$ of V such that $G[A]$ is K_3-free and $G[B]$ is P_3-free (that is, A induces a disjoint union of cliques). A graph is in-partitionable if it is not partitionable.

It is known that recognizing partitionable graphs is NP-complete [49]. On the other hand, a result of Stacho [109, Theorems 7.7 and 7.8] shows that recognizing partitionable graphs can be done in polynomial-time when restricted to the class of chordal graphs. Further, as recognizing partitionable graphs can be expressed in monadic second order logic without edge set quantification, the problem can be efficiently solved on graphs with bounded treewidth [36] or bounded clique-width [37]. We extend these results as follows.

Theorem 4.3. *Recognizing partitionable graphs is* NP*-complete even when restricted to the following classes:*

(1) planar graphs,

(2) K_4-free graphs,

(3) bull-free graphs,

(4) perfect graphs.

The problem of recognizing partitionable graphs restricted to the class of cographs can be done in polynomial-time as can be seen in our next theorem.

Theorem 4.4. *A cograph G is partitionable if and only if G does not contain the graphs $H_1, H_2, \ldots, H_{17}$ illustrated in Figure 4.1.*

Indeed, for every fixed graphs G and H, it is well-known that recognizing if H is an induced graph of G can be done in constant time. Hence, since the forbidden list in Theorem 4.4 has finite size, Theorem 4.4 immediately yields a polynomial-time algorithm for the class of cographs.

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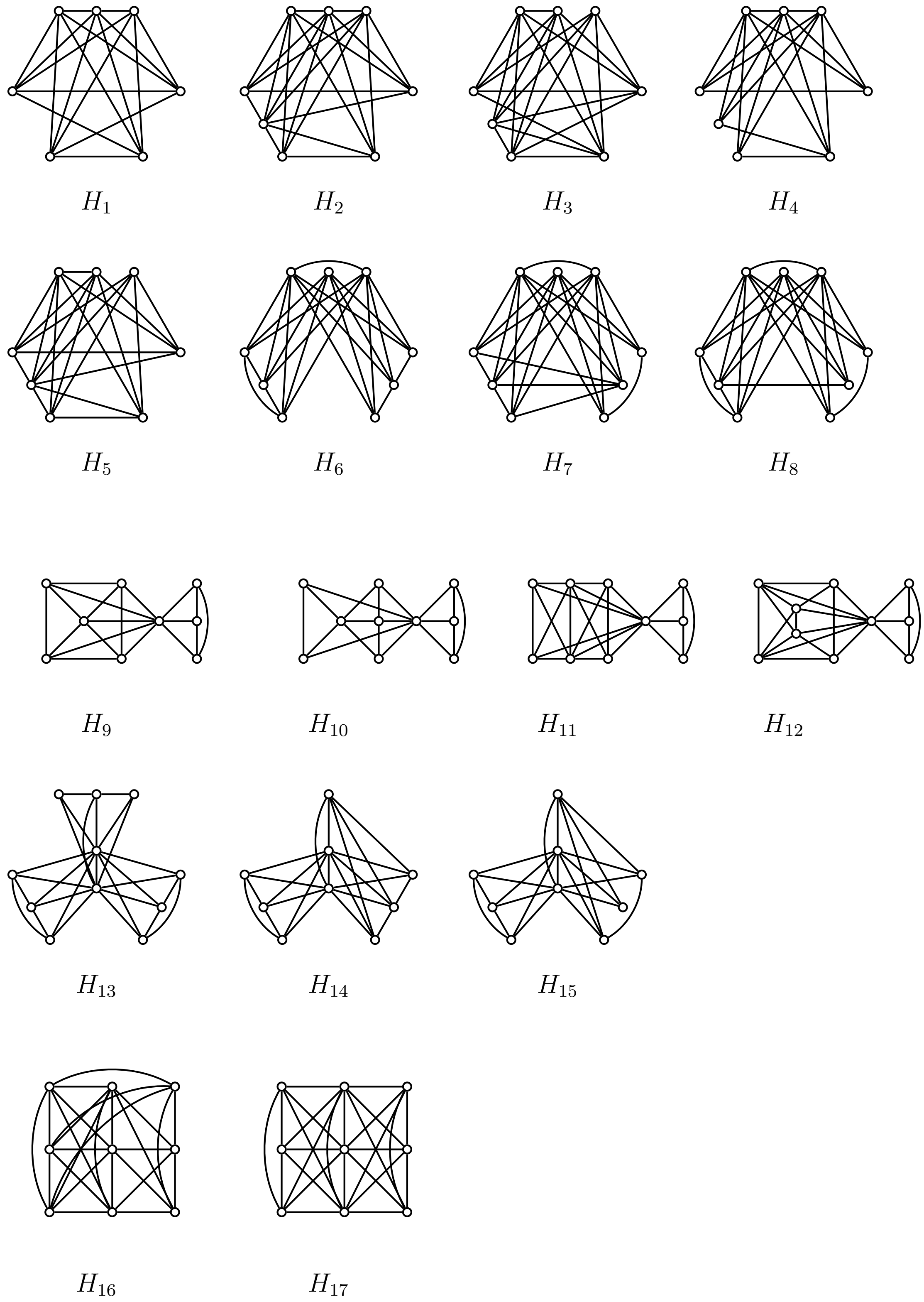

Figure 4.1: Forbidden subgraphs of partitionable cographs.

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The reader may wonder if Theorem 4.3(3) settles Problem 4.1. We state without proof that the reduction graph in our construction for the bull-free case (see Section 4.3) violates Definition 4.2(3) and therefore leaves Problem 4.1 yet unresolved.

We end this section with a brief discussion of some of the related work by noting that the class of partitionable graphs generalizes the following classes of *monopolar graphs* and $(1,2)$-*partitionable graphs*:

- A graph G is monopolar if there exists a partition $\{A, B\}$ of V such that $G[A]$ is P_3-free and $G[B]$ is a stable set. Monopolar graphs have been extensively studied in recent years. Indeed, deciding if a graph is monopolar is NP-complete [49] even when restricted to triangle-free graphs [33] and planar graphs [86]; in contrast, the problem is tractable on several graph classes such as the classes of cographs [45], polar permutation graphs [43], chordal graphs [44], line graphs [32] and several others [34].

- A graph is (k,l)-*partitionable* if it can be partitioned in up to k cliques and l independent sets with $k + l \geq 1$. Table 4.1 contains trivial complexity results on (k,l)-partitionable problems in special classes of graphs for $k + l \leq 2$. In [40] efficient algorithms are devised for solving the (k,l)-partition problem on cographs, where k and l are finite. In [39] a characterization of (k,l)-partitionable cographs by forbidden induced subgraphs is provided, where k and l are finite. These results were later extended to P_4-*sparse graphs* [22] (a graph is P_4-sparse if every set of five vertices of the graph induce at most one P_4). and extended P_4-*laden graphs* [23] (a graph is extended P_4-laden if every induced subgraph with at most six vertices that contains more than two induced P_4's is $\{2K_2, C_4\}$-free).

4.2 Preliminaries

Let G and H be two vertex disjoint graphs. The (complete) join of G and H, denoted $G \oplus H$, is obtained by joining every vertex of G to every vertex of H. An *odd hole* is an induced cycle of odd length at least 5. An *odd antihole* is the complement of an

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k	l	graph class	recognition	forbidden cographs	forbiden others
0	1	edge-less	$\mathcal{O}(n)$	K_2	none
1	0	complete	$\mathcal{O}(n+m)$	$2K_1$	none
1	1	split	$\mathcal{O}(n+m)$	$2K_2, C_4$	C_5
0	2	bipartite	$\mathcal{O}(n+m)$	K_3	odd cycles
2	0	co-bipartite	$\mathcal{O}(n+m)$	$3K_1$	odd co-cycles

Table 4.1: Some trivial complexity results on (k,l)-partitionable problems.

odd hole. Recall that a graph G is perfect if for every induced subgraph H of G the chromatic number of H equals the order of the largest clique of H. By the Strong Perfect Graph Theorem [31], a graph is perfect if and only if it contains no odd hole and no odd antihole. A bull is a self-complementary graph with degree sequence $(3,3,2,1,1)$; an illustration of bull is given in Figure 4.2. Recall that a graph is planar if it can be drawn in the plane such that no two of its edges cross. The class of *cographs* is equivalent to the class of P_4-free graphs [35] and a cograph or its complement is disconnected unless the cograph is K_1. A P_3-free graph is equivalent to a (disjoint) union of cliques and a $\overline{P_3}$-free graph is equivalent to a (complete) join of stable sets. Split graphs are exactly the $(1,1)$-partitionable graphs; they are characterized by the absence of $2K_2$, C_4 and C_5. The intersection of cographs and split graphs are the threshold graphs, characterized by the absence of $2K_2$, C_4 and P_4. The diamond, paw, and butterfly graph can be expressed as $K_2 \oplus 2K_1$, $K_1 \oplus (K_1 \cup K_2)$ and $K_1 \oplus 2K_2$, respectively. The k-wheel graph is obtained from a cycle C of order $k-1$ by joining a vertex not in C to every vertex of C.

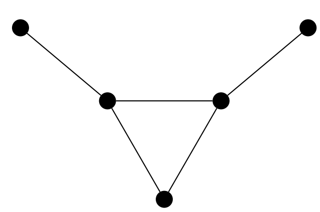

Figure 4.2: The bull graph

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4.3 Hardness Results

In this section we prove Theorem 4.3. Firstly we provide some gadgets that we will use in reductions from 3SAT. Let $G = (V, E)$ be a graph and let $\{A, B\}$ be a partition of V such that A induces a K_3-free subgraph of G and B induces a P_3-free subgraph of G. For short we write that a vertex $v \in V$ is *red* if it belongs to A and *blue* if it belongs to B. Recall that a partition is (unless stated otherwise) a partition of V into red and blue vertices.

4.3.1 Negators

A graph with two designated vertices x and y is a *blue negator* if it has no partition where both x and y are blue, but admits a partition where at most one of the vertices x and y is blue and the blue vertex has no blue neighbour. In what follows we implicitly use this partition. Examples of blue negators are given in Figure 4.3.

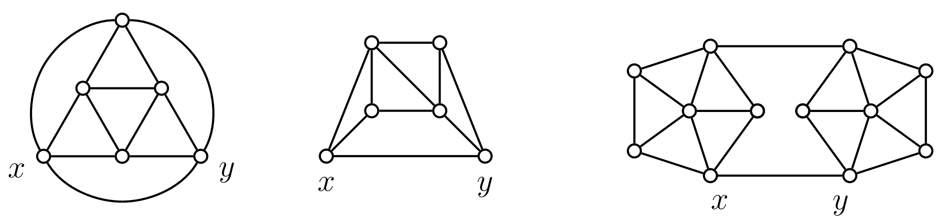

Figure 4.3: blue negators: the octahedron, the P_6^2-component and the two-wheel.

Similarly, a graph with two designated vertices x and y is a *red negator* if it has no partition where both x and y are red, but admits a partition where at most one of the vertices x and y is red and the blue vertex has no blue neighbour. In what follows we implicitly use this partition. Examples of red negators are given in Figure 4.4.

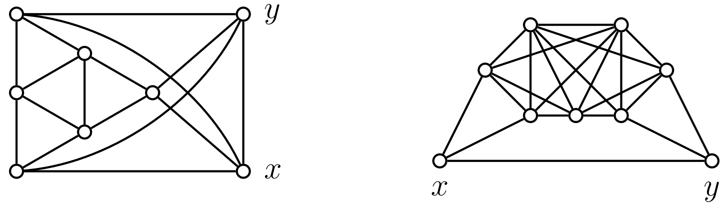

Figure 4.4: red negators: the sun component (left) and the bull-free component (right).

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Finally, a *strong negator* is a graph that is both a red negator and a blue negator. Examples of strong negators, built from red or blue negators, are shown in Figure 4.5.

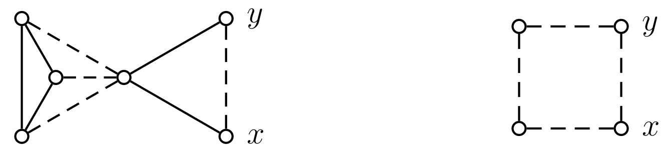

Figure 4.5: Strong negators: the dashed lines represent blue negators in the left graph and red negators in the right. Their endpoints are the vertices x and y from these negators.

4.3.2 Reduction from 3SAT

We can now describe a generic reduction from 3SAT to our partition problem. Let φ be an instance of 3SAT, that is, a propositional formula in CNF with clauses $c_1, c_2, \ldots c_m$. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the variables that occur in φ. We may safely assume that a variable and its negation do not occur in the same clause, a variable does not occur more than once in the same clause, and a variable occurs in at least two clauses. For every variable $x_i \in X$ we create a *truth assignment component* which is a ladder, whose edges are replaced by red or strong negators, with m rungs $x_{i,1}y_{i,1}, x_{i,2}y_{i,2}, \ldots, x_{i,m}y_{i,m}$, such that $\{x_{i,j} \mid 1 \leq j \leq m\}$ and $\{y_{i,j} \mid 1 \leq j \leq m\}$ become independent sets in the truth assignment component. Note that the vertices x and y from the red or strong negators that form the ladder uniquely partition into two subsets, each of which can be either red or blue, see Figure 4.6. For every clause c_j we create a *satisfaction test component* which is a P_3. For every literal x_i that appears in clause c_j we identify the vertex $x_{i,j}$ of the truth assignment component for x_i with a vertex of the satisfaction test component for c_j, and the vertex $y_{i,j}$ of the truth assignment component is identified with a vertex from the satisfaction test component if $\neg x_i$ appears in c_j. This completes the construction of the reduction graph Q.

Lemma 4.5. *The graph Q is partitionable if and only if φ is satisfiable.*

Proof. If φ is satisfiable we fix a satisfying truth assignment of the variables in X. All true literals become red and all false literals become blue. Hence every

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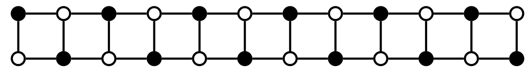

Figure 4.6: A ladder with twelve rungs. In every partition, all black vertices belong to one part and all white vertices belong to the other part.

truth assignment component is partitionable and every satisfaction test component contains at least one red vertex and thus at most two (possibly adjacent) blue vertices with no other blue neighbours. This implies Q is partitionable.

Conversely, suppose Q be partitionable. We assign the boolean value true to each variable x_i with red vertices representing the literal x_i and blue vertices representing $\neg x_i$, and false if the roles are the other way around. This defines a consistent truth assignment for all variables in X because each truth assignment component is a ladder with at least two rungs. Consider a clause c_j of φ. It corresponds to a satisfaction test component of Q which is a P_3. Hence at least one vertex of this satisfaction test component is red and therefore c_j is satisfied. ■

Planar Graphs

To show the NP-completeness of the partition problem restricted to planar graphs we reduce instead from the NP-complete problem PLANAR-3SAT [87]. We use the planar strong negators depicted in Figure 4.5 whose dashed lines are blue negators from Figure 4.3. The fact that Q is planar can be easily derived from [87]: it suffices to contract every edge between a truth assignment component and a satisfaction test component to obtain the associated (planar) graph of an instance of planar 3SAT.

K_4-free Graphs

The partition problem becomes trivial when restricted to triangle-free graphs (these are graphs that do not contain K_3 as induced subgraph) because all vertices can be made red. Restricted to K_4-free graphs the problem remains NP-complete: the sun component in Figure 4.4 can be used in the generic reduction from 3SAT.

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Bull-free Graphs

The partition problem remains NP-complete when restricted to bull-free graphs: the graph Q is bull-free if the bull-free component from Figure 4.4 is used in the generic reduction from 3SAT.

Perfect Graphs

Here we show NP-completeness of the partition problem when restricted to perfect graphs. A new reduction is required for that purpose. Firstly we provide some gadgets that will be used in the reduction.

Gadgets

We use the P_6^2-component as the blue negator gadget shown in Figure 4.3 and the strong negator gadget shown at the left of Figure 4.5.

The *literal gadget* is shown in Figure 4.7 where the double line symbolises the strong negator gadget. The gadget is partitionable and for every partition it has at least two blue endpoints.

The *propagator gadget* is shown in Figure 4.7. The gadget is partitionable and for every partition it has exactly one or three blue endpoints.

Together the literal gadget and the propagator gadget form the satisfaction test component.

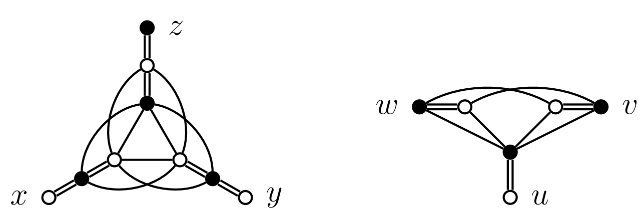

Figure 4.7: The literal gadget with endpoints x, y, z and the propagator gadget with endpoints u, v, w along with a partition where the white vertices are in the P_3-free part and the black vertices are in K_3-free part. Note that the propagator gadget is not symmetric.

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Reduction from Positive-1-in-3-SAT

We describe a reduction from POSITIVE-1-IN-3-SAT, which is known to be NP-hard [106], to our partition problem on perfect graphs. An instance of POSITIVE-1-IN-3-SAT is a set of variables $X = \{x_1, x_2, \ldots, x_n\}$ and a set of clauses $C = \{c_j \mid i = 1, 2, \ldots, m\}$, such that each $c_j = (l_{j,1} \vee l_{j,2} \vee l_{j,3})$ consists of three positive literals and each literal $l_{j,k}$ is x_i for some $x_i \in X$. The problem is to determine whether there exists a truth assignment to the variables in X such that $\varphi = c_1 \wedge c_2 \wedge \cdots \wedge c_m$ is satisfiable with exactly one true literal per clause.

(Note that the first steps of our construction are identical to the construction described earlier. We include them for the convenience of the reader.)

For every variable $x_i \in X$ we create a truth assignment component which is a ladder, whose edges are strong negators, with m rungs $x_{i,1}y_{i,1}, x_{i,2}y_{i,2}, \ldots, x_{i,m}y_{i,m}$, such that the set $\{x_{i,j} \mid 1 \leq j \leq m\}$ of *literal vertices* and the set $\{y_{i,j} \mid 1 \leq j \leq m\}$ of *propagator vertices* become independent sets in the truth assignment component. Note that the vertices x and y from the strong negators that form the ladder uniquely partition into two subsets, each of which can be either red or blue, see Figure 4.6. For a clause $c = (x_1 \vee x_2 \vee x_3)$ where x_1, x_2 and x_3 are the ith, jth and kth occurrence, respectively, create a copy H_c of the literal gadget whose endpoints are identified with literal vertices $x_{1,i}, x_{2,j}$ and $x_{3,k}$, and a copy R_c of the propagator gadget whose endpoints are identified with propagator vertices $y_{1,i}, y_{1,j}$ and $y_{2,k}$. H_c and R_c are said to be the *literal gadget* and *propagator gadget*, respectively, of C. This completes the construction of the reduction graph L.

Lemma 4.6. *The graph L is partitionable if and only if φ is satisfiable with exactly one true literal per clause.*

Proof. If φ is satisfiable with exactly one true literal per clause we fix a satisfying truth assignment of the variables in X. All literal vertices corresponding to true literals become red and all literals vertices corresponding to false literals become blue. This implies that every truth assignment component is partitionable. For a literal gadget H_c and a propagator gadget R_c of clause C, H_c has two blue endpoints and R_c has one blue endpoint. Thus H_c and R_c are partitionable and hence L is

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partitionable.

Conversely, suppose L be partitionable. We assign boolean value true to each variable x_i with red vertices representing the literal x_i and false otherwise. Consider the literal gadget H_c and propagator gadget R_c of clause C. If for contradiction all endpoints of H_c are blue then all endpoints of R_c become red, a contradiction. Hence exactly one endpoint of H_c is red and therefore C has exactly one true literal. ■

The next lemma in conjunction with the above lemma completes the proof of Theorem 4.3(5).

Lemma 4.7. *The graph L is perfect.*

Proof. We first prove that L contains no odd hole. The next two properties follow by a careful examination of L.

(1) The gadgets and truth assignment component are odd hole-free

(2) Each induced path between the endpoints of a literal or propagator gadget has even length

Let C be an induced cycle of length at least 4 in L. By (1), if C is an induced subgraph of a gadget or truth assignment component then C has even length. Otherwise, let $R_1, \ldots, R_k$ be induced subgraphs of truth assignment components occurring on C in that cyclic order. Clearly there exists a 2-colouring ϕ of $R_1 \cup \cdots \cup R_k$ where colour class 1 are literal vertices and colour class 2 are propagator vertices. Observe that the segment P_i of C joining R_i and R_{i+1} is a path contained in a literal or propagator gadget whose endpoints are endpoints of that gadget. Since the endpoints of P_i have the same colour under ϕ and P_i has even length by (2), ϕ can be extended to a 2-colouring that includes P_i. This implies G contains no odd hole.

It remains to show that L contains no odd antihole. We already established that L does not contain $\overline{C_5} = C_5$. Moreover, L is K_5-free and hence $\overline{C_{2k+1}}$-free, $k \geq 5$. Now K_4 is contained in $\overline{C_7}$ (and hence $\overline{C_9}$). The only occurrences of K_4 in L are in a literal or strong negator gadget. By considering adjacencies between such a K_4 and the rest of the graph it can be verified that L does not contain $\overline{C_7}$ and $\overline{C_9}$. ■

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4.4 Cographs

In this section we prove Theorem 4.4. We start by characterizing subclasses of partitionable cographs by forbidden induced subgraphs. These results will be useful in establishing the main theorem.

4.4.1 Subclasses of Partitionable Cographs

A set of definitions and lemmas is initially required.

Definition 4.8. A bi-threshold graph is a bipartite or threshold graph.

Definition 4.9. A monopolar nearly split graph is a monopolar or $(1,2)$-partitionable graph.

Lemma 4.10. *Let G be a graph. If G contains P_3 and K_3, then G contains $F_1 = P_3 \cup K_3$, $F_2 = \text{diamond}$, or $F_3 = \text{paw}$.*

Proof. Consider the triangle. If there is a vertex with exactly one or two neighbours in the triangle we have F_3 or F_2, respectively. If two non-adjacent vertices with three neighbours in the triangle exist we have F_2. If none of these cases applies to any triangle in G then all triangles form a clique with no neighbours in the rest of the graph. Consequently we find F_1. ■

Lemma 4.11. *Let G be a cograph. If G contains P_3 and $2K_2$, then G contains $Q_1 = P_3 \cup K_2$, or $Q_2 = \text{butterfly}$.*

Proof. Consider the disjoint edges e_1 and e_2 in $2K_2$. Let G_1 be the component that contains e_1. First suppose G_1 contains e_2. Let v be a vertex adjacent to some endpoint of e_1 and on a path between e_1 and e_2. Since G is a cograph any induced path between two vertices in a component of G has length at most 2. As e_1 and e_2 have no edges between them every induced path between e_1 and e_2 has length 2. It follows that v must be adjacent to every vertex in e_1 and e_2, in which case Q_2 is found. Finally suppose G_1 does not contain e_2. If there is a vertex with exactly one neighbour in e_1 then Q_1 is obtained. If this case does not apply to any vertex in G_1 then G_1 forms a clique with no neighbours in the rest of the graph and Q_1 is again obtained. ■

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Lemma 4.12. *Let G be a cograph. If G is C_4-free and contains $P_3, 2K_2$ and K_3, then G contains $S_1 = F_1$, $S_2 = Q_2$, $S_3 = K_2 \cup \text{paw}$, or $S_4 = K_2 \cup \text{diamond}$.*

Proof. Consider the disjoint edges e_1 and e_2 in $2K_2$. Let G_1 be the component containing e_1. If G_1 contains e_2 then, by the same argument as in the proof of Lemma 4.11, we find S_2. So suppose G_1 does not contain e_2. We distinguish a number of cases. If there exists two non-adjacent vertices with two neighbours in e_1 then S_4 is obtained. If there exists two non-adjacent vertices with one and two neighbours, respectively, in e_1 then S_3 is obtained. If there exists two adjacent vertices with one and two neighbours, respectively, in e_1 then S_4 is found. If none of these cases applies to any edge in G_1 then, by considering the absence of P_4 and C_4, G_1 either forms (i) a star graph with no neighbours in the rest of the graph, or (ii) a clique with no neighbours in the rest of the graph. In the case of (i) we find S_1. In the case of (ii) if G_1 contains a triangle then S_1 is obtained and if G_1 is a single edge we find S_1, S_3 or S_4, by Lemma 4.10. ■

Lemma 4.13. *Let G be a cograph. If G contains P_3 and $2K_3$, then G contains $W_1 = 2K_3 \cup P_3$, $W_2 = K_3 \cup \text{diamond}$, $W_3 = K_3 \cup \text{paw}$, or $W_4 = K_1 \oplus 2K_3$.*

Proof. Consider the disjoint triangles t_1 and t_2 in $2K_3$. If t_1 and t_2 share a neighbour then, by considering the absence of P_4, W_4 is obtained. Otherwise, by a similar argument to that in Lemma 4.10, we find W_1, W_2, or W_3. ■

Bi-threshold Cographs

This section establishes the following theorem.

Theorem 4.14. *Let G be a connected cograph. Then G is bi-threshold if and only if G does not contain the graphs $B_1, \ldots, B_6$ depicted in Figure 4.8.*

(1) $B_1 = \text{butterfly}$.

(2) $B_2 = C_4 \oplus K_1$.

(3) $B_3 = 2K_1 \oplus (K_2 \cup K_1)$.

(4) $B_4 = K_2 \cup \text{diamond}$.

(5) $B_5 = K_3 \cup P_3$.

(6) $B_6 = K_2 \cup \text{paw}$.

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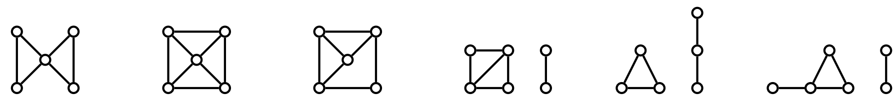

B_1 B_2 B_3 B_4 B_5 B_6

Figure 4.8: The graphs $B_1, \ldots, B_6$.

Proof. ($\Leftarrow$) Recall that a threshold graph is $(C_4, P_4, 2K_2)$-free and a bipartite graph is triangle-free. But the graphs $B_1, \ldots, B_6$ each contain a triangle, and C_4 or $2K_2$.

($\Rightarrow$) Let G be a connected cograph that is neither bipartite nor threshold and vertex minimal. If G is complete the result is easily seen to be true. So suppose that G contains P_3. In particular G must contain K_3, and C_4 or $2K_2$. We distinguish two cases.

Case 1: G contains C_4.

Since G is connected and P_4-free there exists a triangle and a quadrangle that share an edge. The third vertex of the triangle has another neighbour in the quadrangle, otherwise there would be a P_4. Consequently G contains B_2 or B_3.

Case 2: G contains $2K_2$.

By Lemma 4.12, G contains B_1, B_4, B_5 or B_6. This completes the proof. ∎

Monopolar Cographs

In [45] a forbidden induced subgraph characterization of monopolar cographs, defined in the paper as (s,k)-polar cographs where $\min(s,k) \leq 1$, is presented. (Note that our definition of monopolar graphs is different). Essentially, the same proof shows the following result.

Theorem 4.15. *Let G be a connected cograph. Then G is monopolar if and only if G has no induced subgraph isomorphic to the graphs $J_1, \ldots, J_4$ depicted in Figure 4.9.*

(1) $J_1 = 5 -$ wheel.

(2) $J_2 = K_1 \oplus (P_3 \cup K_2)$.

(3) $J_3 = K_2 \oplus 2K_2$.

(4) $J_4 = (K_2 \cup K_1) \oplus (K_2 \cup K_1)$.

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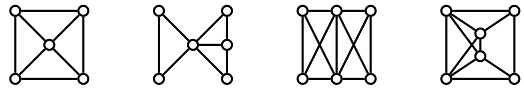

J_1 J_2 J_3 J_4

Figure 4.9: The graphs J_1, J_2, J_3, J_4.

Proof. ($\Leftarrow$) Recall that a monopolar graph is a graph that can be partitioned into an independent set and a union of cliques. Since every J_i is not a union of cliques, it must contain a join of stable sets in any partition. It is routine to verify that there exists no partition of these graphs such that their join of stable sets in the partition is a stable set.

($\Rightarrow$) Since G is connected it is the join of two cographs $G[A]$ and $G[B]$. Since a threshold graph is $(C_4, P_4, 2K_2)$-free, it suffices to consider the following cases.

Case 1: $G[A]$ is not a threshold graph.

Subcase 1.1: $G[A]$ contains C_4.

Since $G[B]$ is non-empty, G contains J_1.

Subcase 1.2: $G[A]$ contains $2K_2$.

If $G[B]$ contains K_2 then G contains J_3. So suppose $G[B]$ is a stable set. If $G[A]$ contains P_3 then, by Lemma 4.11, $G[A]$ contains Q_1 or Q_2. If $G[A]$ contains Q_2 then G contains $J_3 = Q_2 \oplus K_1$, and if $G[A]$ contains Q_1 then G contains $J_2 = Q_1 \oplus K_1$. Finally if $G[A]$ is P_3-free then $G = G[A] \oplus G[B]$ is monopolar. This completes Case 1.

It may be assumed by symmetry that both $G[A]$ and $G[B]$ do not contain C_4, $2K_2$ and P_4 and hence form threshold graphs.

Case 2: $G[A]$ and $G[B]$ are threshold graphs.

Subcase 2.1: $G[A]$ contains a triangle.

(1) If $G[A]$ is a clique then $G[B]$ being a threshold graph, G is also a threshold graph and therefore monopolar.

(2) Suppose $G[A]$ contains a paw or a diamond. In both cases $G[A]$ contains P_3. If $G[B]$ contains $2K_1$ then G contains $J_1 = P_3 \oplus 2K_1$, and if $G[B]$ is a clique then G

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is a threshold graph.

(3) Suppose $G[A]$ contains at least one isolated vertex besides the triangle. If $G[B]$ contains P_3 then G contains $J_1 = P_3 \oplus 2K_1$. So we may assume that $G[B]$ is a union of cliques. If $G[B]$ contains $K_2 \cup K_1$ then G contains $J_4 = (K_2 \cup K_1) \oplus (K_2 \cup K_1)$. If $G[B]$ is a non-trivial stable set then G is monopolar. Finally if $G[B]$ is a clique then G forms a threshold graph.

Subcase 2.2: Both $G[A]$ and $G[B]$ are triangle-free.

(1) Suppose $G[A]$ contains P_3. If $G[B]$ contains $2K_1$ then G contains $J_1 = P_3 \oplus 2K_1$. If $G[B]$ is a clique then G is a threshold graph.

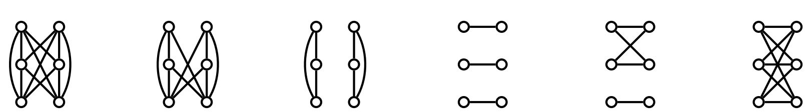
(2) We may thus assume, by symmetry, that $G[A]$ and $G[B]$ are P_3-free. First suppose $G[A]$ contains $K_2 \cup K_1$. If $G[B]$ contains $K_2 \cup K_1$ then G contains $J_4 = (K_2 \cup K_1) \oplus (K_2 \cup K_1)$. So let $G[B]$ be $(K_2 \cup K_1)$-free. If $G[B]$ is a stable set then G is monopolar. Otherwise, $G[B]$ is a clique in which case G is a threshold graph. Second suppose $G[A]$ is a clique. Since $G[B]$ is a threshold graph, it follows that G is a threshold graph. Finally if $G[A]$ is a stable set, $G[B]$ being P_3-free it follows that G is monopolar. This completes the proof. ■

Remark 4.16. The graphs J_1, J_2, J_3 and J_4 are $(1,2)$-partitionable connected cographs.

Proof. If $C(J_i)$ denotes a maximum clique of J_i, $i = 1, \ldots, 4$, then $J_i[V \setminus C(J_i)]$ is bipartite. ■

Monopolar Nearly Split Cographs

In this section we characterize monopolar nearly split cographs by a finite list of forbidden induced subgraphs. First we need an auxiliary result.



D_1 D_2 D_3 $\overline{D_1}$ $\overline{D_2}$ $\overline{D_3}$

Figure 4.10: The graphs D_1, D_2, D_3 and their complements.

Proposition 4.17 ([40])**.** *A cograph is $(2,1)$-partitionable if and only if it does not contain the graphs $\overline{D_1}$, $\overline{D_2}$, $\overline{D_3}$ depicted in Figure 4.10.*

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Corollary 4.18. *A cograph is* $(1,2)$*-partitionable if and only if it does not contain the graphs* $D_1 = \overline{3K_2}, D_2 = 2K_2 \oplus 2K_1, D_3 = 2K_3$ *depicted in Figure 4.10.*

We are now ready to prove the theorem.

Theorem 4.19. *Let* G *be a connected cograph. Then* G *is a monopolar nearly split graph if and only if* G *does not contain the graphs* $R_1, \ldots, R_8$ *depicted in Figure 4.11.*

(1) $R_1 = 2K_1 \oplus 2K_1 \oplus 2K_1$.

(2) $R_2 = 2K_2 \oplus (K_2 \cup K_1)$.

(3) $R_3 = 2K_1 \oplus (P_3 \cup K_2)$.

(4) $R_4 = K_1 \oplus (2K_1 \oplus 2K_2)$.

(5) $R_5 = K_2 \oplus 2K_3$.

(5') $R_5 = K_1 \oplus (K_1 \oplus 2K_3)$.

(6) $R_6 = K_1 \oplus (P_3 \cup 2K_3)$.

(7) $R_7 = K_1 \oplus (K_3 \cup (P_3 \oplus K_1))$.

(8) $R_8 = K_1 \oplus (K_3 \cup (K_1 \oplus (K_1 \cup K_2)))$.

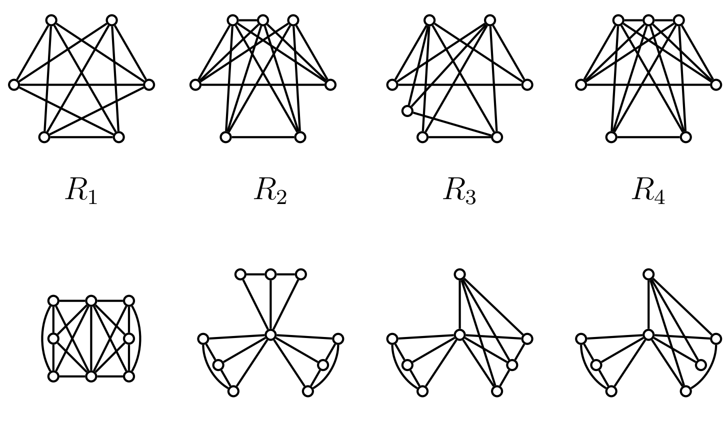

Figure 4.11: The graphs $R_1, \ldots, R_8$.

Proof. ($\Leftarrow$) This is proved by a careful case analysis.

($\Rightarrow$) Suppose G is neither monopolar nor $(1,2)$-partitionable and vertex minimal. Since G is connected it is the join of two cographs $G[A]$ and $G[B]$. By the minimality of G, $G[A]$ and $G[B]$ are either monopolar or $(1,2)$-partitionable. We distinguish a number of cases.

Case 1: $G[A]$ and $G[B]$ are $(K_2 \cup K_1)$-free.

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It follows that G is a join of stable sets. Hence G either contains $R_1 = \overline{3K_2}$ or is $(1,2)$-partitionable.

Case 2: $G[A]$ and $G[B]$ contain $K_2 \cup K_1$.

(1) If $G[A]$ contains C_4 then G contains $R_1 = C_4 \oplus 2K_1$.
(2) If $G[A]$ contains $2K_2$ then G contains $R_2 = 2K_2 \oplus (K_2 \cup K_1)$.
(3) By symmetry if $G[A]$ and $G[B]$ are threshold graphs then G is $(1,2)$-partitionable.

Case 3: $G[A]$ is $(K_2 \cup K_1)$-free, and $G[B]$ contains $K_2 \cup K_1$.

Subcase 3.1: $G[A]$ is a clique.

If $G[B]$ is $(1,2)$-partitionable then G is $(1,2)$-partitionable. Otherwise, $G[B]$ must be monopolar. By Corollary 4.18 and given that $J_1 \subset D_1$ it follows that $G[B]$ contains D_2 or D_3.
(1) If $G[B]$ contains D_2 then G contains $R_4 = K_1 \oplus D_2$.
(2) Suppose $G[B]$ contains D_3. If $G[A]$ has at least 2 vertices then G contains $R_5 = K_2 \oplus D_3$. So suppose $G[A]$ is a single vertex. If $G[B]$ is P_3-free then G is monopolar. If $G[B]$ contains P_3 then, by Lemma 4.13, $G[B]$ contains W_1, W_2, W_3 or W_4. It follows that G contains $R_6 = K_1 \oplus W_1$, $R_7 = K_1 \oplus W_2$, $R_8 = K_1 \oplus W_3$, or $R_5 = K_1 \oplus W_4$, respectively.

Subcase 3.2: $G[A]$ is an independent set.

The case where $G[A]$ is a single vertex is covered in Subcase 3.1. We may thus assume that $G[A]$ contains $2K_1$. If $G[B]$ is P_3-free then G is monopolar. If $G[B]$ is a threshold graph then G is $(1,2)$-partitionable. Otherwise, $G[B]$ contains C_4, or P_3 and $2K_2$. If $G[B]$ contains C_4 then G contains $R_1 = 2K_1 \oplus C_4$. If $G[B]$ contains P_3 and $2K_2$ then, by Lemma 4.11, $G[B]$ contains Q_1 or Q_2. Hence G contains $R_3 = 2K_1 \oplus Q_1$ or $R_4 = 2K_1 \oplus Q_2$, respectively.

Subcase 3.3: $G[A]$ contains $2K_1 \oplus 2K_1$.

Since $G[B]$ contains $K_2 \cup K_1$, it follows that G contains $R_1 = 2K_1 \oplus 2K_1 \oplus 2K_1$.

Subcase 3.4: $G[A] = qK_1 \oplus K_r$ for some integers $q \geq 2$ and $r \geq 1$.

If $G[B]$ is a threshold graph then G is $(1,2)$-partitionable. Otherwise, $G[B]$ contains

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$2K_2$ or C_4. It follows that G either contains R_4 or R_1, respectively. This completes the proof. ■

4.4.2 The Proof of Theorem 4.4

Before proving Theorem 4.4, we need the following auxiliary results. The first lemma is implicit in [45].

Lemma 4.20. *Minimal in-partitionable cographs are connected.*

Proof. Let G be a cograph. Suppose to the contrary that G is disconnected and vertex minimal in-partitionable. Let $\{A, B\}$ be a partition of V such that $G = G[A] \cup G[B]$. By the minimality of G, $G[A]$ and $G[B]$ are partitionable. Let C and D be a partition of $G[A]$, P and Q a partition of $G[B]$ such that $G[C]$, $G[P]$ are bipartite, and $G[D]$, $G[Q]$ are P_3-free. It follows that $G[C \cup P]$ is bipartite and $G[D \cup Q]$ is P_3-free, which is a partition of G. ■

Lemma 4.21. *Let G be a cograph, and let $\{A, B\}$ be a partition of V such that $G = G[A] \oplus G[B]$. If both $G[A]$ and $G[B]$ are threshold graphs then G is partitionable.*

Proof. Let $G' = G[A]$ and $G'' = G[B]$. Let $\{C, D\}$ be a partition of $V(G')$ such that C induces a clique and D induces a stable set. Similarly, let $\{F, P\}$ be a partition of $V(G'')$ such that F induces a clique and G induces a stable set. Since $G = G[A] \oplus G[B]$, it follows that $G[C \cup F] = G[C] \oplus G[F]$ is a clique and $G[D \cup P] = G[D] \oplus G[P]$ is a complete bipartite graph. ■

The following graphs depicted in Figure 4.1 will be used:

(1) $H_1 = 2K_1 \oplus 2K_1 \oplus 2K_1 \oplus K_1$

(2) $H_2 = P_3 \oplus K_1 \oplus 2K_2$

(3) $H_3 = 2K_1 \oplus (K_2 \cup K_1) \oplus (K_2 \cup K_1)$

(4) $H_4 = P_3 \oplus (K_2 \cup P_3)$

(5) $H_5 = (K_2 \cup K_1) \oplus K_1 \oplus 2K_2$

(6) $H_6 = (K_2 \cup K_1) \oplus (K_3 \cup P_3)$

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(7) $H_7 = (K_2 \cup K_1) \oplus (K_2 \cup (P_3 \oplus K_1))$

(8) $H_8 = (K_2 \cup K_1) \oplus (K_2 \cup (K_1 \oplus (K_2 \cup K_1)))$

(9) $H_9 = K_1 \oplus (K_3 \cup (C_4 \oplus K_1))$

(10) $H_{10} = K_1 \oplus (K_3 \cup (K_1 \oplus (P_3 \cup K_2)))$

(11) $H_{11} = K_1 \oplus (K_3 \cup (K_2 \oplus 2K_2))$

(12) $H_{12} = K_1 \oplus (K_3 \cup ((K_2 \cup K_1) \oplus (K_2 \cup K_1)))$

(13) $H_{13} = K_2 \oplus (P_3 \cup 2K_3)$

(14) $H_{14} = K_2 \oplus (K_3 \cup (P_3 \oplus K_1))$

(15) $H_{15} = K_2 \oplus (K_3 \cup (K_1 \oplus (K_1 \cup K_2))$

(16) $H_{16} = (K_3 \cup K_2) \oplus (K_3 \cup K_1)$

(17) $H_{17} = K_3 \oplus 2K_3$

We are now ready to prove the theorem.

Proof of Theorem 4.4. ($\Leftarrow$) This follows by a careful case analysis.

($\Rightarrow$) Suppose G is vertex minimal in-partitionable. By Lemma 4.20, G is connected. We prove that G must contain one of the graphs $H_1, \ldots, H_{17}$. We make use of the fact that since a cograph contains no odd hole, G is partitionable if and only if there exists a partition $\{X, Y\}$ of V such that X induces a P_3-free graph and Y induces a bipartite graph.

Claim 3. *If G has no universal vertex then G contains one of the graphs $H_1, \ldots, H_8$, H_{16}.*

Proof of Claim 3. Since G is connected it is the join of two cographs $G[A]$ and $G[B]$. By the minimality of G, $G[A]$ and $G[B]$ are partitionable. Since G has no universal vertex, $G[A]$ and $G[B]$ have no universal vertex. Consequently $G[A]$ and $G[B]$ each contain $2K_1$. We consider a number of cases.

Case 1: $G[A]$ is P_3-free.

$G[A]$ is a union of at least two cliques C_1, C_2 because it contains $2K_1$.

Subcase 1.1: $G[B]$ is P_3-free.

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Similarly $G[B]$ is a union of at least two cliques C_3, C_4. If $G[B]$ or $G[A]$ is bipartite then G is partitionable. So it may be assumed, without loss of generality, that $|C_1|, |C_3| \geq 3$. Moreover C_2 or C_4 contains K_2, otherwise $G[A]$ and $G[B]$ form threshold graphs and G is partitionable by Lemma 4.21. Hence G contains $H_{16} = (K_3 \cup K_2) \oplus (K_3 \cup K_1)$.

Subcase 1.2: $G[B]$ contains P_3.

(1) $G[A]$ is a stable set of order at least two.

If $G[B]$ is monopolar then G is partitionable. Otherwise, by Theorem 4.15, $G[B]$ contains one of the graphs J_1, J_2, J_3, J_4. It follows that G contains $H_1 = 2K_1 \oplus J_1$, $H_2 = 2K_1 \oplus J_3$, $H_3 = 2K_1 \oplus J_4$, or $H_4 = 2K_1 \oplus J_2$, respectively.

(2) $G[A] = K_r \cup K_1$ for some integer $r \geq 2$.

If $G[B]$ is a threshold graph then G is $(1, 2)$-partitionable. If $G[B]$ is bipartite then G is partitionable. If, on the other hand, $G[B]$ contains K_3, and C_4 or $2K_2$ then, by Theorem 4.14, $G[B]$ contains one of the graphs B_1, B_2, B_3, B_4, B_5 or B_6. It follows that G contains $H_5 = (K_2 \cup K_1) \oplus B_1$, $H_1 = 2K_1 \oplus B_2$, $H_3 = (K_2 \cup K_1) \oplus B_3$, $H_7 = (K_2 \cup K_1) \oplus B_4$, $H_6 = (K_2 \cup K_1) \oplus B_5$, or $H_8 = (K_2 \cup K_1) \oplus B_6$, respectively.

(3) $G[A]$ contains $2K_2$.

If $G[B]$ is bipartite then G is partitionable. We may thus assume that $G[B]$ contains a triangle (as $G[B]$ is a cograph). Since $G[B]$ contains P_3, by Lemma 4.10, $G[B]$ contains one of the graphs F_1, F_2, F_3. It follows that G contains $H_6 = (K_2 \cup K_1) \oplus F_1$, $H_2 = 2K_2 \oplus F_2$, or $H_5 = 2K_2 \oplus F_3$, respectively. This completes Case 1.

Case 2: $G[A]$ and $G[B]$ contain P_3.

Since G is a cograph, it has no induced C_5. Together with the fact that a threshold graph is a $(C_4, P_4, 2K_2)$-free graph it suffices to consider the following cases.

Subcase 2.1: $G[A]$ contains C_4.

Then G contains $H_1 = C_4 \oplus P_3$.

Subcase 2.2: $G[A]$ contains $2K_2$.

By Lemma 4.11, $G[A]$ contains Q_1 or Q_2. It follows that G contains $H_4 = P_3 \oplus Q_1$ or $H_2 = P_3 \oplus Q_2$, respectively.

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Subcase 2.3: $G[A]$ and $G[B]$ are threshold graphs.

It follows by Lemma 4.21 that G is partitionable. This completes Case 2 and the proof of Claim 5.6. ■

Claim 4. *If G has a universal vertex v such that $G' = G \setminus v$ is disconnected then G contains one of the graphs $H_9, H_{10}, H_{11}, H_{12}$.*

Proof of Claim 4. Let $r \geq 2$ be an integer and $\mathcal{G} = \{G_1, \ldots, G_r\}$ be the set of components of G'. By the minimality of G for every graph $G_i \in \mathcal{G}$ the graphs G_i and $G'_i = v \oplus G_i$ are partitionable. We claim that there exists a graph $T \in \mathcal{G}$ that is $(1, 2)$-partitionable, but not monopolar.

To see this, consider a graph $K \in \mathcal{G}$. If every partition of K into k disjoint cliques and l independent sets has $\min(k, l) \geq 2$ then $K' = v \oplus K$ is in-partitionable. So we may assume that each $G_i \in \mathcal{G}$ is either $(1, 2)$-partitionable or monopolar. But If every $G_i \in \mathcal{G}$ is monopolar then G'_i admits a partition where v is in the bipartite part. Hence, as the G_i's are disjoint, G also admits a partition where v is again in the bipartite part.

From now on, let $G_j \in \mathcal{G}$ be a graph that is $(1, 2)$-partitionable, but not monopolar for some $j \in \{1, \ldots r\}$. By Theorem 4.15 and Remark 4.16, G_j contains one of the graphs J_1, J_2, J_3, J_4. For contradiction suppose there exists no $p \neq j$ such that G_p contains K_3. Let $C(G_j)$ and $S(G_j)$ denote the partition of G_j into a clique and a bipartite graph, respectively. Then $V = A \cup B$ where $A = v \cup C(G_j)$, and $B = S(G_j) \cup \bigcup_{p \neq j} G_p$ is a partition of V where $G[A]$ is P_3-free and $G[B]$ is bipartite, a contradiction.

We conclude that G contains $H_9 = v \oplus (K_3 \cup J_1)$, $H_{10} = v \oplus (K_3 \cup J_2)$, $H_{11} = v \oplus (K_3 \cup J_3)$ or $H_{12} = v \oplus (K_3 \cup J_4)$. ■

Claim 5. *If G has a universal vertex v such that $G' = G \setminus v$ is connected then G contains one of the graphs $H_1, H_2, H_4, H_5, H_{13}, H_{14}, H_{15}, H_{17}$.*

Proof of Claim 5. By the minimality of G, G' is partitionable. In particular, G' is neither monopolar nor $(1, 2)$-partitionable, otherwise $G = G' \oplus v$ is partitionable. Hence, by Theorem 4.19, G' contains one of the graphs $R_1, \ldots, R_8$. It follows that

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G contains $H_1 = v \oplus R_1$, $H_5 = v \oplus R_2$, $H_4 = v \oplus R_3$, $H_2 = v \oplus R_4$, $H_{17} = v \oplus R_5$, $H_{13} = v \oplus R_6$, $H_{14} = v \oplus R_7$, or $H_{15} = v \oplus R_8$. This completes the proof of Claim **??** and Theorem 4.4. ■

4.5 Final Remarks

Improvements of Theorem 4.3 were recently obtained by Bouguеret and Ochem [19]. They showed that the problem remains NP-complete for the intersection of any two classes amongst the first three classes considered in Theorem 4.3. They also showed NP-completeness for other small classes, such that graphs with maximum degree 4, line graphs, and $(C_1, \ldots, C_t)$-free graphs for any fixed $t \geq 5$.

A possible extension of our result on cographs is the following. Given a finite sequence $(H_1, \ldots, H_k)$ of cographs, can we compute the finite set F of cographs such that for every cograph G, the vertices of G can be partitioned into $V_1, \ldots, V_k$ such that $G[V_i]$ is H_i-free if and only if G is F-free? A celebrated result of Damaschke [38] states that, given a graph property $\mathcal{P}$ that is closed under taking induced subgraphs, the class of cographs $G \in \mathcal{P}$ can be characterized by a finite list of forbidden induced subgraphs. Therefore we know that such a finite set F of forbidden induced subgraphs exists. It would be enough to prove a recursive bound on the size of the graphs in F. Note that for $k = 2$, $H_1 = K_3$ and $H_2 = P_3$ we described the set F in Section 4.4. We remark that this question has been fully settled in [50] in the situation where, for each $1 \leq i \leq k$, V_i either induces a clique or an independent set, and, for any two V_j, V_h where $j \neq h$, each vertex vertex of V_j is joined to each vertex of V_h.

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Chapter 5

Erdős-Ko-Rado Theorems for a Family of Trees

5.1 Introduction

In this chapter we consider graph theoretic versions of the following famous result due to Erdős, Ko and Rado. The extremal case is characterized by Hilton and Milner [70].

EKR Theorem (Erdős, Ko, Rado [47]; Hilton, Milner [70]) *Let n and r be positive integers, $n \geq r$, let S be a set of size n and let $\mathcal{A}$ be a family of subsets of S each of size r that are pairwise intersecting. If $n \geq 2r$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

Moreover, if $n > 2r$ the upper bound is attained only if the sets in $\mathcal{A}$ contain a fixed element of S.

Let $K_{1,n}$ denote a claw. Let $\mu(G)$ denote the minimum size of a maximal independent set in G.

Given a graph G and an integer $r \geq 1$, let $\mathcal{I}^{(r)}(G)$ denote the family of independent sets of G of cardinality r. For a vertex v of G, let $\mathcal{I}_v^{(r)}(G)$ be the subset of $\mathcal{I}^{(r)}(G)$ containing all sets that contain v. This is called an *r-star* (or just star) and v is its *centre*. We say that G is r-EKR if no pairwise intersecting family $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$

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is larger than the biggest r-star, and strictly r-EKR if every pairwise intersecting family that is not an r-star is smaller than the the largest r-star of $\mathcal{I}^{(r)}(G)$.

The EKR Theorem can be seen as a statement about the maximum size of a family of pairwise intersecting independent sets of size r in the empty graph on n vertices. We quickly obtain another formulation of the EKR Theorem by noting that an independent set of the claw that contains more than one vertex contains only leaves.

Theorem 5.1. *Let n and r be positive integers, $n \geq r$. The claw $K_{1,n}$ is r-EKR if $n \geq 2r$ and strictly r-EKR if $n > 2r$.*

There exist EKR results for several graph classes. We briefly summarize some of the related work and refer instead the reader to [16] for an excellent exposition. One of the first results of this kind is due to Berge [4].

Theorem 5.2 ([4]). *Let $r \geq 1$, $t \geq 2$ and G be the disjoint union of r copies of K_t. Then G is r-EKR.*

The extremal case of this result was addressed by Livingstone [88] and other proofs were also given in [66, 100]. Some generalizations of Theorem 5.2 can also be found in [6, 46, 59]. We state one such generalization due to Holroyd, Spencer and Talbot [71].

Theorem 5.3 ([71]). *If G is the disjoint union of $n \geq r$ complete graphs each of order at least two, then G is r-EKR.*

Let us state one more result obtained in [71].

Theorem 5.4 ([71]). *If G is the disjoint union of $n \geq 2r$ complete graphs, cycles and paths, and an isolated vertex, then G is r-EKR.*

Theorem 5.4 was subsequently generalized by Woodroofe [118] who showed that if G is the disjoint union of n arbitrary graphs including an isolated vertex, then G is r-EKR if $n \geq 2r$.

Let us remark on the importance of having an isolated vertex in results of this kind. Having a largest r-star that is invariant under some graph operations makes

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it easier to show that a graph is r-EKR via an induction argument. The presence of an isolated vertex in a graph allows us to systematically locate a largest r-star having such a property. Indeed, consider a graph H obtained from a graph G by some graph operations such that u belongs to and is isolated in H. For any vertex $v \in V(H) \setminus \{u\}$, it suffices to describe an injection f from $\mathcal{I}_v^{(r)}(H)$ into $\mathcal{I}_u^{(r)}(H)$: for $A \in \mathcal{I}_v^{(r)}(H)$, let $f(A) = A$ if $u \in A$; otherwise, let $f(A) = A \cup \{u\} \setminus \{v\}$.

The kth *power* of a graph G on n vertices, denoted G_n^k, is constructed by joining any two vertices that are joined by a path on k or fewer edges in G.

Theorem 5.5 ([71]). *If $k, n, r \geq 1$, then P_n^k is r-EKR.*

Two proofs of Theorem 5.5 were given in [71]. We note that the first proof relied on Talbot's [110] remarkable and complicated proof that the kth power of a cycle on n vertices is r-EKR for $k, n, r \geq 1$ (no simpler proof of this result has been found to date).

In [72] Holroyd and Talbot proved that if G is the disjoint union of two complete multipartite graphs, then G is r-EKR if $2r \leq \mu(G)$. Further, mindful of earlier results, they made the following conjecture.

Conjecture 5.6 ([72]). *Let r be a positive integer and let G be a graph. Then G is r-EKR if $\mu(G) \geq 2r$ and strictly r-EKR if $\mu(G) > 2r$.*

This conjecture appears difficult to prove or disprove. The reader is referred to [17, 69, 68, 72, 73] for further examples confirming Conjecture 5.6 on a number of graph classes.

The aim of this chapter is to contribute to the conjecture for the class of trees. Before we state our results let us briefly mention the related work. As mentioned before, a usual technique to prove EKR results is to find the centre of the largest r-star of a graph. Thus Hurlbert and Kamat [73] made the following conjecture.

Conjecture 5.7 ([73]). *Let n and r be positive integers, $n \geq r$. If T is a tree on n vertices, then there is a largest r-star of T whose centre is a leaf.*

They were able to confirm Conjecture 5.7 whenever $1 \leq r \leq 4$ [73]. The conjecture does not, however, hold for $r \geq 5$ as shown by Baber [2].

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5.1.1 Results

We consider a subfamily of trees called *elongated claws*. An elongated claw has one vertex that is its *root*. Every other vertex has degree 1 or 2 (it is possible that the root also has degree 1 or 2). In other words, an elongated claw is a graph obtained from a claw by subdividing each edge zero or more times. A vertex of degree 1 is called a *leaf*. A path from the root to a leaf is a *limb*. A limb is *short* if it contains only one edge. If every leaf is distance 2 from the root (that is, if every limb contains two edges), then the graph is a *depth-two claw*.

We are now ready to state our main results.

Theorem 5.8. *Let r be a positive integer and let G be a depth-two claw such that $\mu(G) \geq 2r$. Then G is strictly r-EKR if $\mu(G) \geq 2r - 1$.*

Theorem 5.8 confirms (and is stronger than) Conjecture 5.6 for depth-two claws.

Theorem 5.9. *Let n and r be positive integers, $n \geq 2r$, and let G be an elongated claw with n leaves and at least one short limb. Then G is r-EKR.*

Theorem 5.9 does not confirm (but only supports) Conjecture 5.6 for the class of elongated claws with short limbs since $\mu(G)$ may be much larger than the number of leaves in G.

We remark that similar EKR results (that is, with weaker bounds than that of Conjecture 5.6) were obtained in [71, Theorem 8] and [118, Proposition 4.3]. Satisfying the bound of Conjecture 5.6 in Theorem 5.9, and in general for elongated claws, is left as an open problem. In particular, the reader is referred to Section 5.4 for a brief discussion on our investigation towards EKR results for other elongated claws.

In the immediately following two sections we prove Theorems 5.8 and 5.9.

5.2 Depth-two Claws

The following lemma is useful in the proofs of both Theorem 5.8 and Theorem 5.9.

Lemma 5.10. *Let r be a positive integer, and let G be an elongated claw. Then there is a largest r-star of G whose centre is a leaf.*

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Proof. Let v be a vertex of G that is not a leaf, and let L be the limb of G that contains v (if v is the root, then L can be any limb). Let x be the leaf of L. We find an injection f from $\mathcal{I}_v^{(r)}(G)$ to $\mathcal{I}_x^{(r)}(G)$ which proves that $|\mathcal{I}_x^{(r)}(G)| \geq |\mathcal{I}_v^{(r)}(G)|$ and the lemma immediately follows.

Let w be the unique neighbour of x. Let $A \in \mathcal{I}_v^{(r)}(G)$.

1. If $x \in A$, then let $f(A) = A$.

2. If $x \notin A$ and $w \notin A$, then let $f(A) = A \backslash \{v\} \cup \{x\}$.

3. If $x \notin A$ and $w \in A$, then let $X = \{x = x_1, x_2, \ldots, x_m = v\}$ be the set of vertices in L from x towards v. Let $A \cap X = \{x_{i_1}, \ldots, x_{i_j}\} = Y$ for some $m > j \geq 1$. Let $Z = \{x_{i_1 - 1}, \ldots, x_{i_j - 1}\}$. Observe that $|Y| = |Z|$ and $x \in Z$ since $w \in Y$. Then let $f(A) = (A \cup Z) \backslash Y$.

To prove that f is injective we consider distinct $A_1, A_2 \in \mathcal{I}_v^{(r)}(G)$. If $f(A_1)$ and $f(A_2)$ are defined by the same case (of the three above), then it is clear that $f(A_1)$ and $f(A_2)$ are distinct. When they are defined by different cases, we simply note that in the first $f(A)$ always contains v, in the second $f(A)$ contains neither v nor any of its neighbours, and in the third $f(A)$ contains a neighbour of v. ■

We note that Lemma 5.10 confirms Conjecture 5.7 for elongated claws.

Remark. The property of elongated claws in Lemma 5.10 is a much weaker version of the *degree sort property*; a graph has this property if the size of an r-star centred at u is at least the size of an r-star centred at v whenever the degree of u is less than that of v. Hurlbert and Kamat [73] observed that depth-two claws have this property. We note that not all elongated claws possess it. For example, consider an elongated claw with three limbs of lengths 1, 2 and 3. Then the 4-star centred at the neighbour of the root in the limb of length 3 has size 2, but the 4-star centred at the leaf of the limb of length 2 has size 1. It remains to determine which elongated claws — or, more generally, which trees — have the degree sort property. We might also ask which trees have the following weaker property: if $i < j$, then the size of the largest r-star of all those stars centred at vertices of degree i is at least the size of the largest r-star of all those centred at vertices of degree j.

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Lemma 5.11. *Let n and r be positive integers, $n \geq r$, and let G be a depth-two claw with n leaves. Then the size of the largest r-star of G is*

$$\binom{n-1}{r-1}2^{r-1} + \binom{n-1}{r-2}.$$

Proof. By Lemma 5.10, there is a largest r-star whose centre is a leaf (and clearly, by symmetry, all leaves are equivalent). So let v be a leaf of G and let c be the root of G. Define a partition: $\mathcal{B} = \{B \in \mathcal{I}_v^{(r)}(G) : c \notin B\}$ and $\mathcal{C} = \{C \in \mathcal{I}_v^{(r)}(G) : c \in C\}$. Then $|\mathcal{B}| = \binom{n-1}{r-1}2^{r-1}$ since each member of $\mathcal{B}$ intersects $r-1$ of the $n-1$ limbs that do not contain v and can contain either of the 2 vertices (other than the root) of each of those limbs. And $|\mathcal{C}| = \binom{n-1}{r-2}$ since each member of $\mathcal{C}$ contains $r-2$ of the $n-1$ leaves other than v. The proof is complete. ■

In order to prove Theorem 5.8, we shall need two auxiliary results.

Theorem 5.12 (Meyer [93]; Deza and Frankl [41])**.** *Let n, r and t be positive integers, $n \geq r$, $t \geq 2$, and let G be the disjoint union of n copies of K_t. Then G is r-EKR and strictly r-EKR unless $r = n$ and $t = 2$.*

For a family of sets $\mathcal{A}$ and nonnegative integer s, the s-shadow of $\mathcal{A}$, denoted $\partial_s\mathcal{A}$, is the family $\partial_s\mathcal{A} = \{S : |S| = s, \exists A \in \mathcal{A}, S \subseteq A\}$.

Lemma 5.13 (Katona [82])**.** *Let a and b be nonnegative integers and let $\mathcal{A}$ be a family of sets of size a such that $|A \cap A'| \geq b \geq 0$ for all $A, A' \in \mathcal{A}$. Then $|\mathcal{A}| \leq |\partial_{a-b}\mathcal{A}|$*

The proof of Theorem 5.8 is inspired by a proof of the EKR theorem [60]. To the best of our knowledge, the proof is the first to make use of shadows in the context of graphs.

Proof of Theorem 5.8. Let c be the root of G and let n be the number of leaves of G. Note that $n = \mu(G)$ so $n \geq 2r-1$. Let $\mathcal{A} \subseteq I^{(r)}(G)$ be any pairwise intersecting family. Define a partition $\mathcal{B} = \{A \in \mathcal{A} : c \notin A\}$ and $\mathcal{C} = \{A \in \mathcal{A} : c \in A\}$.

Notice that each vertex in each member of $\mathcal{B}$ is either a leaf or the neighbour of a leaf. For $B \in \mathcal{B}$, let M_B be the set of r leaves that each either belongs to B or is adjacent to a vertex in B. We say that M_B *represents* B. Let $\mathcal{M} = \{M_B : B \in \mathcal{B}\}$.

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Note that each member of $\mathcal{M}$ might represent many different members of $\mathcal{B}$. In fact, consider $M \in \mathcal{M}$. It can represent any independent set that, for each leaf $\ell \in M$, contains either ℓ or its unique neighbour. There are 2^r such sets, but they can be partitioned into complementary pairs so, as $\mathcal{B}$ is pairwise intersecting, the number s_M of members of $\mathcal{B}$ that M represents is at most 2^{r-1}. We also note that $\mathcal{M}$ is pairwise intersecting (since $\mathcal{B}$ is pairwise intersecting).

We have that

$$|\mathcal{B}| = \sum_{M \in \mathcal{M}} s_M \leq \binom{n-1}{r-1} 2^{r-1}, \tag{5.2.1}$$

where the inequality follows from Theorem 5.12.

For $B \in \mathcal{B}$, let N_B be the set of $n-r$ leaves that neither belong to B nor are adjacent to a vertex in B. Notice that M_B and N_B partition the set of leaves. Let $\mathcal{N} = \{N_B : B \in \mathcal{B}\}$. For any pair $B_1, B_2 \in \mathcal{B}$, we know that M_{B_1} and M_{B_2} intersect, so $|M_{B_1} \cup M_{B_2}| \leq 2r-1$. The leaves not in this union are members of both N_{B_1} and N_{B_2} and there are at least $n-(2r-1) \geq 0$ of them. Thus we can apply Lemma 5.13 to $\mathcal{N}$ with $a = n-r$, $b = n-(2r-1)$ to obtain

$$|\mathcal{N}| \leq |\partial_{r-1}\mathcal{N}|. \tag{5.2.2}$$

Notice that, by definition, $\partial_{r-1}\mathcal{N}$ is a collection of sets of $r-1$ leaves each of which is, for some $B \in \mathcal{B}$, a subset of N_B and is therefore disjoint to M_B and so certainly does not intersect B.

Let us try to bound the size of $\mathcal{C}$. Each $C \in \mathcal{C}$ contains a distinct set of $r-1$ leaves. We know this set must intersect every member of $\mathcal{B}$ so it cannot be a member $\partial_{r-1}\mathcal{N}$. Thus we find

$$|\mathcal{C}| \leq \binom{n}{r-1} - |\partial_{r-1}\mathcal{N}|. \tag{5.2.3}$$

We apply (5.2.2) to (5.2.3) and note that $|\mathcal{N}| = |\mathcal{M}|$ to obtain

$$|\mathcal{C}| \leq \binom{n}{r-1} - |\mathcal{M}|. \tag{5.2.4}$$

Since $s_M \leq 2^{r-1}$ for each $M \in \mathcal{M}$, equality holds in (5.2.1) only if $|\mathcal{M}| \geq \binom{n-1}{r-1}$.

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Thus combining (5.2.1) and (5.2.4):

$$\begin{aligned} |\mathcal{A}| &= |\mathcal{B}| + |\mathcal{C}| \\ &\leq \sum_{M \in \mathcal{M}} s_M + \binom{n}{r-1} - |\mathcal{M}| \\ &\leq \binom{n-1}{r-1} 2^{r-1} + \binom{n}{r-1} - \binom{n-1}{r-1} \\ &= \binom{n-1}{r-1} 2^{r-1} + \binom{n-1}{r-2}. \end{aligned} \tag{5.2.5}$$

This proves that G is r-EKR by Lemma 5.10. We now show that G is strictly r-EKR. If $r = n$, then $r = 1$ so the result trivially holds. Suppose $r < n$. Then, by Theorem 5.12, equality holds in (5.2.1) and therefore in (5.2.5) only if $\mathcal{B}$ is an r-star centred at a leaf x or a neighbour y of a leaf. It follows easily that $\mathcal{C} = \emptyset$ if $\mathcal{A} = \mathcal{I}_y^{(r)}(G)$; thus $\mathcal{A} = \mathcal{I}_x^{(r)}(G)$ as desired. ■

Remark. We demonstrate that if G is a depth-two claw with n leaves, then G is not n-EKR by describing a pairwise intersecting family that is larger than the largest n-star. Let c be the root of G and let $G' = G - c$, a graph containing n copies of K_2 each of which contains one leaf of G. Clearly G' contains 2^n independent sets of size n which can be partitioned into complementary pairs. Let $\mathcal{B}$ be a family of 2^{n-1} independent sets of size n formed by considering each complementary pair and choosing either the one that contains the greater number of leaves of G, or, if they each contain half the leaves, choosing one arbitrarily. Notice that $\mathcal{B}$ is pairwise intersecting, but is not a star. Let $\mathcal{C} = \{C \in \mathcal{I}^{(n)}(G) : c \in C\}$. Clearly, $|\mathcal{C}| = \binom{n}{n-1} = n$ and for each pair $C \in \mathcal{C}$, $B \in \mathcal{B}$, we have that $C \cap B \neq \emptyset$. Thus if $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$, then $\mathcal{A}$ is pairwise intersecting, maximal and $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| = 2^{n-1} + n$. By Lemma 5.11, $\mathcal{A}$ has one more element than the largest n-star in G.

The above remark motivates the following conjecture.

Conjecture 5.14. *Let n and r be positive integers, $n > r$ and let G be a depth-two claw with n leaves. Then G is r-EKR.*

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5.3 Elongated Claws with Short Limbs

In this section we will prove Theorem 5.9. We require some terminology and lemmas. For a vertex v of a graph G, let $G - v$ denote the graph obtained by deleting v and incident edges from G, and let $G\downarrow v$ be the graph obtained from G by deleting the vertex v and all its neighbours and their incident edges.

The following lemma has essentially the same proof as Lemma 2.5 in [73], but we include a proof for completeness.

Lemma 5.15. *Let r be a positive integer, and let G be a graph. Let v be a vertex of G and let u be a vertex of $G\downarrow v$. Then*

$$|\mathcal{I}_u^{(r)}(G)| = |\mathcal{I}_u^{(r)}(G - v)| + |\mathcal{I}_u^{(r-1)}(G\downarrow v)|.$$

Proof. Define a partition of $\mathcal{I}_u^{(r)}(G)$: $\mathcal{B} = \{A \in \mathcal{I}_u^r(G) : v \notin A\}$ and $\mathcal{C} = \{A \in \mathcal{I}_u^{(r)}(G) : v \in A\}$. Observe that $|\mathcal{B}| = |\mathcal{I}_u^{(r)}(G - v)|$ and $|\mathcal{C}| = |\mathcal{I}_u^{(r-1)}(G\downarrow v)|$. This implies the lemma. ■

Lemma 5.16. *Let r be a positive integer and let G be an elongated claw with a short limb with root c. If x is a leaf of G adjacent to c, then x is the centre of a largest r-star of G.*

Proof. Let v be a vertex in G that is not a leaf adjacent to c. We must show that $\mathcal{I}_v^{(r)}(G)$ is no larger than $\mathcal{I}_x^{(r)}(G)$. If $v = c$ this is immediate since $\{A \setminus \{c\} \cup \{x\} : A \in \mathcal{I}_c^{(r)}(G)\}$ has the same cardinality as $\mathcal{I}_c^{(r)}(G)$ and is a subset of $\mathcal{I}_x^{(r)}(G)$.

If $v \neq c$, let L be the limb of G that contains v. To prove the lemma, we find an injection f from $\mathcal{I}_v^{(r)}(G)$ to $\mathcal{I}_x^{(r)}(G)$. Let $A \in \mathcal{I}_v^{(r)}(G)$. We distinguish a number of cases.

1. If $x \in A$, then $f(A) = A$.

2. If $x \notin A$ and $c \notin A$, then $f(A) = A\setminus\{v\} \cup \{x\}$.

3. If $x \notin A$ and $c \in A$, let $X = \{v = x_1, \ldots, x_m\}$ be the set of vertices from v towards the neighbour x_m of c in L. Let $Y = A \cap X = \{x_{i_1}, \ldots, x_{i_j}\}$ for some $m > j \geq 1$. Let $Z = \{x_{i_1+1}, \ldots, x_{i_j+1}\}$ and observe that $|Y| = |Z|$. Then $f(A) = (A \cup Z \cup \{x\})\setminus(Y \cup \{c\})$.

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It can be verified that f is injective as required. ■

We now prove Theorem 5.9 using an approach based on that of the proof of [73, Theorem 1.22].

Proof of Theorem 5.9. Let c be the root of G. Let $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$ be any pairwise intersecting family. We must show that $\mathcal{A}$ is no larger than the largest r-star. We use induction on r. If $r = 1$ the result is true so suppose that $r \geq 2$ and that the result is true for smaller values of r.

We now use induction on the number of vertices in G. The base case is that G contains only the root and n leaves; that is, $G = K_{1,n}$ and so the result follows from Theorem 5.1. So suppose that the number of vertices is at least $n + 2$ and that the result is true for graphs with fewer vertices.

Let x be a leaf adjacent to c. Let v be a leaf that is not adjacent to c. Let w be the unique neighbour of v and let z denote the other neighbour of w.

Define $f : \mathcal{A} \to \mathcal{I}^{(r)}(G)$ such that for each $A \in \mathcal{A}$

$$f(A) = \begin{cases} A\backslash\{v\} \cup \{w\}, & v \in A, z \notin A, A\backslash\{v\} \cup \{w\} \notin \mathcal{A} \\ A, & \text{otherwise.} \end{cases}$$

Define the families:

$$\mathcal{A}' = \{f(A) : A \in \mathcal{A}\},$$
$$\mathcal{B} = \{A \in \mathcal{A}' : v \notin A\},$$
$$\mathcal{C} = \{A\backslash\{v\} : v \in A, A \in \mathcal{A}'\}.$$

Notice that

$$|\mathcal{A}| = |\mathcal{A}'| = |\mathcal{B}| + |\mathcal{C}|. \tag{5.3.6}$$

Claim 1. *Each of $\mathcal{B}$ and $\mathcal{C}$ is pairwise intersecting.*

Proof. By the definition of f, we can partition $\mathcal{B}$ into $\mathcal{B}_1 = \{B \in \mathcal{B} : B \in \mathcal{A}\}$ and $\mathcal{B}_2 = \{B \in \mathcal{B} : B \setminus \{w\} \cup \{v\} \in \mathcal{A}\}$. Then $\mathcal{B}_1$ is pairwise intersecting (since $\mathcal{A}$ is intersecting) and $\mathcal{B}_2$ is pairwise intersecting as every member contains w. Next consider $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. As B_1 and $B_2\backslash\{w\}\cup\{v\}$ are both in $\mathcal{A}$ they intersect

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and this intersection does not contain v (since it is not in B_1) so is a superset of $B_1 \cap B_2$. So $\mathcal{B}$ is intersecting.

By definition, if $C \in \mathcal{C}$, then $C \cup \{v\}$ is in $\mathcal{A}'$ and, by the definition of f, also in $\mathcal{A}$. Using the definition of f again, we must have that either z is in C, or $C \cup \{w\}$ is in $\mathcal{A}$. Let C_1 and C_2 be two members of $\mathcal{C}$. Then either they both contain z or if one of them, say C_1, does not, then $C_1 \cup \{w\}$ is in $\mathcal{A}$. As $C_2 \cup \{v\}$ is also in $\mathcal{A}$ and $\mathcal{A}$ is intersecting, we have that $C_1 \cup \{w\}$ and $C_2 \cup \{v\}$ must intersect. By the independence of the two sets, this intersection contains neither v nor w and so C_1 and C_2 must intersect. The claim is proved. ■

Note that $G - v$ is an elongated claw with a short limb, fewer vertices than G and with n leaves. We also note that each member of $\mathcal{B}$ contains r vertices of $G - v$ and, by Claim 1, $\mathcal{B}$ is pairwise intersecting. By the induction hypothesis, $G - v$ is r-EKR and so the largest intersecting families are r-stars, and, by Lemma 5.16, $\mathcal{I}_x^{(r)}(G - v)$ is a largest r-star of $G - v$. Hence

$$|\mathcal{B}| \leq |\mathcal{I}_x^{(r)}(G - v)|. \tag{5.3.7}$$

Note that $G \downarrow v$ is an elongated claw with a short limb, fewer vertices than G and with either n or $n - 1$ leaves. We also note that each member of $\mathcal{C}$ contains $r - 1$ vertices of $G \downarrow v$ and, by Claim 1, $\mathcal{C}$ is pairwise intersecting. By the induction hypothesis, $G \downarrow v$ is $(r-1)$-EKR and so the largest intersecting families are $(r-1)$-stars, and, by Lemma 5.16, $\mathcal{I}_x^{(r-1)}(G \downarrow v)$ is a largest $(r-1)$-star of $G \downarrow v$. Hence

$$|\mathcal{C}| \leq |\mathcal{I}_x^{(r-1)}(G \downarrow v)|. \tag{5.3.8}$$

Combining (5.3.6), (5.3.7) and (5.3.8) and applying Lemma 5.15:

$$\begin{aligned} |\mathcal{A}| &= |\mathcal{B}| + |\mathcal{C}| \\ &\leq |\mathcal{I}_x^{(r)}(G - v)| + |\mathcal{I}_x^{(r-1)}(G \downarrow v)| \\ &= |\mathcal{I}_x^{(r)}(G)| \end{aligned}$$

and the theorem is proved. ■

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5.4 Final Remark

Let us make a remark on our investigation towards a result for elongated claws without a short limb, but with a limb of length 2. The main obstacle lied in the change of location of the largest r-star when using an argument by induction. We point the reader to a notable result of Hilton, Holroyd and Spencer [69] concerned with the EKR property of several copies of powers of cycles.

Theorem 5.17 ([69]). *Let $G = C_{n_1}^{k_1} \cup C_{n_2}^{k_2} \cup \cdots \cup C_{n_s}^{k_s}$ be the disjoint union of s powers of cycles one of which is simple. Then G is r-EKR for $r \geq \mu(G)$.*

Theorem 5.17 is of relevant interest since their proof successfully deals with varying star centres. We therefore believe that their approach might serve as a spring board for future research on Conjecture 5.6 restricted to elongated claws and, in general, to trees.

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